MANONMANIAM SUNDARANAR UNIVERSITY

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DIRECTORATE OF DISTANCE AND CONTINUING EDUCATION

II B.Sc. MATHEMATICS

SEMESTER III

CORE V : VECTOR CALCULUS AND APPLICATIONS

Sub. Code: JMMA31

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Core V : VECTOR CALCULUS AND APPLICATIONS (JMMA31)

UNIT	DETAILS
I	Vector point function – Scalar point function – Derivative of a vector and derivative of a sum of vectors – Derivative of a product of a scalar and a vector point functions – Derivative of a scalar product and vector product.
II	The vector operator “del” – The gradient of a scalar point Function – Divergence of a vector – Curl of a vector – solenidal and irrotational vectors – simple applications
III	Laplacian operator, Vector Identities – Line integral – Simple Problems
IV	Surface integral – Volume integral – Applications
V	Gauss divergence Theorem, Stoke’s Theorem, Green’s Theorem in two dimensions – Applications to real life situation

Recommended Text
P. Duraipandian and Laxmi Duraipandian, Vector Analysis, Emerald Publishers, 2005

UNIT I

VECTOR & SCALAR POINT FUNCTIONS

1.1 Differentiation of Vector Functions

Definition : Vector Functions : If for each value of a scalar variable u, there corresponds a vector f, then f is said to be a vector function of the scalar variable u. The vector function is written as f(u).

Eg., The vector $(acosu)\vec{\imath}$+$(b\ sinu)\vec{\jmath} + (bu)\vec{k}$ is a vector function of the scalar variable u.

1.2 Limit of a vector function

A vector $\overrightarrow{v_0}$ is said to be the limit of the vector function $\vec{f}(u)$, as u tends to u_0, if $\lim_{u\to u_0}\left|\vec{f}(u) - \overrightarrow{v_0}\right| = 0$. This limit is written as $\lim_{u\to u_0}\vec{f}(u) = \overrightarrow{v_0}$.

Result 1.1 : If $\vec{f}(u) = f_1(u)\vec{\imath} + f_2(u)\ \vec{\jmath} + f_3(u)\vec{k}$, then $\lim_{u\to u_0}\vec{f}(u) = \{\lim_{u\to u_0} f_1(u)\}\vec{\imath} + \{\lim_{u\to u_0} f_2(u)\}\vec{\jmath} + \{\lim_{u\to u_0} f_3(u)\}\vec{k}$.

Result 1.2 : $\lim_{u\to u_0}[A(u) * B(u)] = \{\lim_{u\to u_0} A(u)\} * \{\lim_{u\to u_0} B(u)\}$, where * denotes either a plus or a minus or a dot or a cross.

1.3 Derivative of a vector function

A vector function $\vec{f}(u)$ is said to be derivable or differentiable with respect to u, if $\lim_{\Delta u\to 0}\frac{\vec{f}(u+\Delta u)-\vec{f}(u)}{\Delta u}$ exists. This limit is called the derivative or differential coefficient of $\vec{f}(u)$ with respect to u and is denoted by $\frac{d\vec{f}}{du}$.

Higher derivatives of $\vec{f}(u)$ are defined in the same manner and written as

$$\frac{d}{du}\left(\frac{d\vec{f}}{du}\right) = \frac{d^2\vec{f}}{du^2}, \frac{d}{du}\left(\frac{d^2\vec{f}}{du^2}\right) = \frac{d^3\vec{f}}{du^3}\ \&\ so\ on.$$

Remark 1.1 : If $\vec{f}(u)$ is a constant vector, then its derivative is a zero vector because $\vec{f}(u + \Delta u) - \vec{f}(u) = 0$.

Remark 1.2 : If $\vec{f}(u+\Delta u)$ is written as $\vec{f}(u)+\Delta\vec{f}$ then $\vec{f}(u+\Delta u)-\vec{f}(u)=\Delta\vec{f}$ and $\frac{d\vec{f}}{du}=\lim_{\Delta u\to 0}\frac{\Delta\vec{f}}{\Delta u}$.

Remark 1.3 : The differential coefficient of $\vec{f}$ with respect to u is the rate of change of $\vec{f}$ with respect to u.

Theorem 1.1 :

(i) ***If $\emptyset$ is a scalar function of u and '$\vec{a}$' a constant vector, then*** $\frac{d(\emptyset\vec{a})}{du}=\vec{a}\frac{d\emptyset}{du}$

(ii) ***If '$\vec{a}$' is also a function of u, then*** $\frac{d(\emptyset\vec{a})}{du}=\frac{d\emptyset}{du}\vec{a}+\emptyset\frac{d\vec{a}}{du}$.

Proof :

(i) We have $\frac{df}{du}=\lim_{\Delta u\to 0}\frac{\Delta f}{\Delta u}$

Now, $\frac{d(\emptyset\vec{a})}{du}=\lim_{\Delta u\to 0}\frac{\Delta(\emptyset\vec{a})}{\Delta u}=\lim_{\Delta u\to 0}\frac{(\emptyset+\Delta\emptyset)\vec{a}-\emptyset a}{\Delta u}=\lim_{\Delta u\to 0}\frac{[(\emptyset+\Delta\emptyset)-\emptyset]\vec{a}}{\Delta u}$

$$=\lim_{\Delta u\to 0}\frac{\Delta\emptyset}{\Delta u}\vec{a}=\frac{d\emptyset}{du}\vec{a}$$

(ii) Now, $\frac{d(\emptyset\vec{a})}{du}=\lim_{\Delta u\to 0}\frac{\Delta(\emptyset\vec{a})}{\Delta u}=\lim_{\Delta u\to 0}\frac{(\emptyset+\Delta\emptyset)(\vec{a}+\Delta\vec{a})-\emptyset\vec{a}}{\Delta u}$

$$=\lim_{\Delta u\to 0}\frac{\emptyset\vec{a}+\emptyset\Delta\vec{a}+\vec{a}\Delta\emptyset+\Delta\emptyset\cdot\Delta\vec{a}-\emptyset a}{\Delta u}$$

$$=\emptyset\lim_{\Delta u\to 0}\frac{\Delta\vec{a}}{\Delta u}+\vec{a}\lim_{\Delta u\to 0}\frac{\Delta\emptyset}{\Delta u}+\lim_{\Delta u\to 0}(\frac{\Delta\emptyset}{\Delta u}\cdot\Delta\vec{a})$$

$$=\emptyset\lim_{\Delta u\to 0}\frac{\Delta\vec{a}}{\Delta u}+\vec{a}\lim_{\Delta u\to 0}\frac{\Delta\emptyset}{\Delta u}+\lim_{\Delta u\to 0}\frac{\Delta\emptyset}{\Delta u}\cdot\lim_{\Delta u\to 0}\Delta\vec{a}$$

$$=\emptyset\lim_{\Delta u\to 0}\frac{\Delta\vec{a}}{\Delta u}+\vec{a}\lim_{\Delta u\to 0}\frac{\Delta\emptyset}{\Delta u}\quad(\text{since, }\lim_{\Delta u\to 0}\Delta\vec{a}=0)$$

Thus, $\frac{d(\emptyset\vec{a})}{du}=\frac{d\emptyset}{du}\vec{a}+\emptyset\frac{d\vec{a}}{du}$.

Theorem 1.2 : If $\vec{A}$ and $\vec{B}$ are functions of scalar variable u, then prove that

$(\boldsymbol{i})\ \frac{d(\vec{A}+\vec{B})}{du}=\frac{d\vec{A}}{du}+\frac{d\vec{B}}{du}$

$(\boldsymbol{ii})\ \frac{d(\vec{A}\cdot\vec{B})}{du}=\frac{d\vec{A}}{du}\cdot\vec{B}+\vec{A}\cdot\frac{d\vec{B}}{du}$ ***and***

$$(iii)\ \frac{d(\vec{A}\times\vec{B})}{du} = \frac{d\vec{A}}{du}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{du}.$$

Proof : (i) $\frac{d(\vec{A}+\vec{B})}{du} = \lim_{\Delta u\to 0}\frac{(\vec{A}+\Delta\vec{A})+(\vec{B}+\Delta\vec{B})-(\vec{A}+\vec{B})}{\Delta u}$

$$= \lim_{\Delta u\to 0}\frac{\vec{A}+\Delta\vec{A}+\vec{B}+\Delta\vec{B}-\vec{A}-\vec{B}}{\Delta u} = \lim_{\Delta u\to 0}\frac{\Delta\vec{A}+\Delta\vec{B}}{\Delta u} = \lim_{\Delta u\to 0}\frac{\Delta\vec{A}}{\Delta u} + \lim_{\Delta u\to 0}\frac{\Delta\vec{B}}{\Delta u}$$

$$= \frac{d\vec{A}}{du} + \frac{d\vec{B}}{du}$$

$$(ii)\quad \frac{d(\vec{A}\cdot\vec{B})}{du} = \lim_{\Delta u\to 0}\frac{(\vec{A}+\Delta\vec{A})\cdot(\vec{B}+\Delta\vec{B})-(\vec{A}\cdot\vec{B})}{\Delta u}$$

$$= \lim_{\Delta u\to 0}\frac{\vec{A}\cdot\vec{B}+\vec{A}\cdot\Delta\vec{B}+\Delta\vec{A}\cdot\vec{B}+\Delta\vec{A}\cdot\Delta\vec{B}-(\vec{A}\cdot\vec{B})}{\Delta u}$$

$$= \lim_{\Delta u\to 0}\vec{A}\cdot\frac{\Delta\vec{B}}{\Delta u} + \lim_{\Delta u\to 0}\frac{\Delta\vec{A}}{\Delta u}\cdot\vec{B} + \lim_{\Delta u\to 0}(\frac{\Delta\vec{A}}{\Delta u}\cdot\Delta\vec{B})$$

$$= \lim_{\Delta u\to 0}\vec{A}\cdot\frac{\Delta\vec{B}}{\Delta u} + \lim_{\Delta u\to 0}\frac{\Delta\vec{A}}{\Delta u}\cdot\vec{B} + \lim_{\Delta u\to 0}\frac{\Delta\vec{A}}{\Delta u}\lim_{\Delta u\to 0}\Delta\vec{B}$$

$$= \lim_{\Delta u\to 0}\vec{A}\cdot\frac{\Delta\vec{B}}{\Delta u} + \lim_{\Delta u\to 0}\frac{\Delta\vec{A}}{\Delta u}\cdot\vec{B}\ \text{(Since, } \lim_{\Delta u\to 0}\Delta\vec{B} = 0).$$

$$= \vec{A}\cdot\lim_{\Delta u\to 0}\frac{\Delta\vec{B}}{\Delta u} + \lim_{\Delta u\to 0}\frac{\Delta\vec{A}}{\Delta u}\cdot\vec{B}$$

$$= \frac{d\vec{A}}{du}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{du}.$$

$$(iii)\quad \frac{d(\vec{A}\times\vec{B})}{du} = \lim_{\Delta u\to 0}\frac{(\vec{A}+\Delta\vec{A})\times(\vec{B}+\Delta\vec{B})-(\vec{A}\times\vec{B})}{\Delta u}$$

$$= \lim_{\Delta u\to 0}\frac{\vec{A}\times\vec{B}+\vec{A}v\Delta\vec{B}+\Delta\vec{A}\times\vec{B}+\Delta\vec{A}\times\Delta\vec{B}-(\vec{A}\times\vec{B})}{\Delta u}$$

$$= \lim_{\Delta u\to 0}\vec{A}\times\frac{\Delta\vec{B}}{\Delta u} + \lim_{\Delta u\to 0}\frac{\Delta\vec{A}}{\Delta u}\times\vec{B} + \lim_{\Delta u\to 0}(\frac{\Delta\vec{A}}{\Delta u}\times\Delta\vec{B})$$

$$= \lim_{\Delta u\to 0}\vec{A}\times\frac{\Delta\vec{B}}{\Delta u} + \lim_{\Delta u\to 0}\frac{\Delta\vec{A}}{\Delta u}\times\vec{B} + \lim_{\Delta u\to 0}\frac{\Delta\vec{A}}{\Delta u}\times\lim_{\Delta u\to 0}\Delta\vec{B}$$

$$= \lim_{\Delta u\to 0}\vec{A}\times\frac{\Delta\vec{B}}{\Delta u} + \lim_{\Delta u\to 0}\frac{\Delta\vec{A}}{\Delta u}\times\vec{B}\ \text{(Since, } \lim_{\Delta u\to 0}\Delta\vec{B} = 0).$$

$$= \vec{A} \times \lim_{\Delta u \to 0} \frac{\Delta \vec{B}}{\Delta u} + \lim_{\Delta u \to 0} \frac{\Delta \vec{A}}{\Delta u} \times \vec{B}$$

$$= \frac{d\vec{A}}{du} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{du}.$$

Remark 1.4 : It is important to note here that differentiation of vector functions is similar to differentiation of scalar functions but for the fact that in results pertaining to cross products the order of the vectors is not to be changed during differentiation. This restriction is due to the anti-commutative nature of vector multiplication.

Remark 1.5 : The above results can also be established by assuming $\vec{A} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$ and $\vec{B} = b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}$. As an illustration we now differentiate $\vec{A} \cdot \vec{B}$.

$$\frac{d}{du}\left(\vec{A} \cdot \vec{B}\right) = \frac{d(a_1b_1 + a_2b_2 + a_3b_3)}{du}$$

$$= \frac{da_1}{du}b_1 + a_1\frac{db_1}{du} + \frac{da_2}{du}b_2 + a_2\frac{db_2}{du} + \frac{da_3}{du}b_3 + a_3\frac{db_3}{du}$$

$$= \left(\frac{da_1}{du}b_1 + \frac{da_2}{du}b_2 + \frac{da_3}{du}b_3\right) + \left(a_1\frac{db_1}{du} + a_2\frac{db_2}{du} + a_3\frac{db_3}{du}\right)$$

$$= \left(\frac{da_1}{du}\vec{\imath} + \frac{da_2}{du}\vec{\jmath} + \frac{da_3}{du}\vec{k}\right) \cdot (b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}) + (a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}) \cdot \left(\frac{db_1}{du}\vec{\imath} + \frac{db_2}{du}\vec{\jmath} + \frac{db_3}{du}\vec{k}\right)$$

$$= \left\{\frac{d(a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k})}{du}\right\} \cdot (b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}) + (a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}) \cdot \left(\frac{d(b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k})}{du}\right)$$

$$= \frac{d\vec{A}}{du} \cdot \vec{B} + \vec{A} \cdot \frac{d\vec{B}}{du}.$$

Theorem 1.3 : If $\vec{A}, \vec{B}, \vec{C}$ are functions of the scalar variable u, then

(i) $\frac{d}{du}\left[\vec{A}, \vec{B}, \vec{C}\right] = \left[\frac{d\vec{A}}{du}, \vec{B}, \vec{C}\right] + \left[\vec{A}, \frac{d\vec{B}}{du}, \vec{C}\right] + \left[\vec{A}, \vec{B}, \frac{d\vec{C}}{du}\right]$

(ii) $\frac{d}{du}\left\{\vec{A} \times \left(\vec{B} \times \vec{C}\right)\right\} = \frac{d\vec{A}}{du} \times \left(\vec{B} \times \vec{C}\right) + \vec{A} \times \left(\frac{d\vec{B}}{du} \times \vec{C}\right) + \vec{A} \times \left(\vec{B} \times \frac{d\vec{C}}{du}\right)$

Proof : To reduce the length we shall use ‘dash’ to denote differentiation.

(i) $\frac{d}{du}\left[\vec{A}, \vec{B}, \vec{C}\right] = \frac{d}{du}\left[\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)\right]$

$$= \left[\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)\right]'$$

$$= \left[\vec{A}' \cdot \left(\vec{B} \times \vec{C}\right)\right] + \vec{A} \cdot \left(\vec{B} \times \vec{C}\right)'$$

$$= \vec{A}' \cdot (\vec{B} \times \vec{C}) + \vec{A} \cdot \{\vec{B} \times \vec{C}' + \vec{B}' \times \vec{C}\}$$

$$= \vec{A}' \cdot (\vec{B} \times \vec{C}) + \vec{A} \cdot (\vec{B} \times \vec{C}') + \vec{A} \cdot \{\vec{B}' \times \vec{C}\}$$

$$= \left[\frac{d\vec{A}}{du}, \vec{B}, \vec{C}\right] + \left[\vec{A}, \frac{d\vec{B}}{du}, \vec{C}\right] + \left[\vec{A}, \vec{B}, \frac{d\vec{C}}{du}\right]$$

(ii) $$\{\vec{A} \times (\vec{B} \times \vec{C})\}' = \{\vec{A}' \times (\vec{B} \times \vec{C})\} + \left\{\vec{A} \times (\vec{B} \times \vec{C})'\right\}$$

$$= \{\vec{A}' \times (\vec{B} \times \vec{C})\} + \vec{A} \times \{(\vec{B} \times \vec{C}') + (\vec{B}' \times \vec{C})\}$$

$$= \{\vec{A}' \times (\vec{B} \times \vec{C})\} + \vec{A} \times (\vec{B} \times \vec{C}') + \vec{A} \times (\vec{B}' \times \vec{C})$$

$$= \frac{d\vec{A}}{du} \times (\vec{B} \times \vec{C}) + \vec{A} \times \left(\frac{d\vec{B}}{du} \times \vec{C}\right) + \vec{A} \times \left(\vec{B} \times \frac{d\vec{C}}{du}\right).$$

1.4 Problems

1. Show that $\frac{d}{du}\left(\vec{A} \times \frac{d\vec{B}}{du} - \frac{d\vec{A}}{du} \times \vec{B}\right) = \vec{A} \times \frac{d^2\vec{B}}{du^2} - \frac{d^2\vec{A}}{du^2} \times \vec{B}$

Solution : $\frac{d}{du}\left(\vec{A} \times \frac{d\vec{B}}{du} - \frac{d\vec{A}}{du} \times \vec{B}\right) = \frac{d}{du}\left(\vec{A} \times \frac{d\vec{B}}{du}\right) - \frac{d}{du}\left(\frac{d\vec{A}}{du} \times \vec{B}\right)$

$$= \frac{d\vec{A}}{du} \times \frac{d\vec{B}}{du} + \vec{A} \times \frac{d^2\vec{B}}{du^2} - \frac{d^2\vec{A}}{du^2} \times \vec{B} - \frac{d\vec{A}}{du} \times \frac{d\vec{B}}{du}$$

$$= \vec{A} \times \frac{d^2\vec{B}}{du^2} - \frac{d^2\vec{A}}{du^2} \times \vec{B}$$

2. Find the derivatives of $\vec{A} \cdot \vec{B}$ and $\vec{A} \times \vec{B}$ with respect to u if $\vec{A} = u^2\vec{i} + u\vec{j} + 2u\vec{k}$ and $\vec{B} = \vec{j} - u\vec{k}$.

Solution : (i) Find $\frac{d}{du}(\vec{A} \cdot \vec{B})$

$$\vec{A} \cdot \vec{B} = (u^2\vec{i} + u\vec{j} + 2u\vec{k}) \cdot (\vec{j} - u\vec{k}) = 0 + u - 2u^2 = u - 2u^2$$

$$\frac{d}{du}(\vec{A} \cdot \vec{B}) = \frac{d}{du}(u - 2u^2) = 1 - 4u.$$

(ii) Find $\frac{d}{du}(\vec{A} \times \vec{B})$

$$\vec{A}\times\vec{B}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ u^2 & u & 2u\\ 0 & 1 & -u\end{vmatrix}$$

$$=\vec{\imath}(-u^2-2u)-\vec{\jmath}(-u^3-0)+\vec{k}(u^2-0)$$

$$=\vec{\imath}(-u^2-2u)+u^3\vec{\jmath}+u^2\vec{k}$$

$$\frac{d}{du}(\vec{A}\times\vec{B}) = (-2u-2)\,\vec{\imath}+3u^2\,\vec{\jmath}+2u\vec{k}$$

3. Find $\frac{d}{du}(\vec{A}\cdot\vec{B})$& $\frac{d}{du}(\vec{A}\times\vec{B})$, if $\vec{A}=\vec{\imath}+u\vec{\jmath}+u^2\vec{k}, \vec{B}=u^2\vec{\imath}-u\vec{\jmath}+\vec{k}$.

[Ans. : $\frac{d}{du}(\vec{A}\cdot\vec{B})=2u, \frac{d}{du}(\vec{A}\times\vec{B})=(1+3u^2)\vec{\imath}+4u^3\vec{\jmath}+(-1-3u^2)\vec{k}$.]

4. Find $\frac{d}{du}(\vec{A}\times\vec{B})$ in the following cases:

(i) $\vec{A}=2u\vec{\imath}+u^2\vec{\jmath},\ \vec{B}=-u\vec{\jmath}+\vec{k}$.

(ii) $\vec{A}=5u^2\vec{\imath}+u\vec{\jmath}+u^3\vec{k}, \vec{B}=sinu\,\vec{\imath}-cosu\vec{\jmath}$

[Ans. (i) $\frac{d}{du}(\vec{A}\times\vec{B})=2u\vec{\imath}-2\vec{\jmath}-4u\vec{k}$.

(ii) $\frac{d}{du}(\vec{A}\times\vec{B})=(-3u^2cosu+u^3sinu)\vec{\imath}-(3u^2sinu+u^3cosu)\vec{\jmath}+(-11ucosu-sinu+5u^2sinu)\vec{k}$]

5. Show that the necessary and sufficient condition for the non-zero vector function $\vec{f}(u)$ to be of constant magnitude is $\vec{f}\cdot\frac{d\vec{f}}{du}=0$. (i.e. $\vec{f}$ *and* $\frac{d\vec{f}}{du}$ are perpendicular to each other).

Solution : Let $\vec{f}=f\hat{f}$, where f is the magnitude of $\vec{f}$. Then $\vec{f}\cdot\vec{f}=f^2$.

Differentiating on both sides w.r.t. u, we get

$$\frac{d\vec{f}}{du}\cdot\vec{f}+\vec{f}\cdot\frac{d\vec{f}}{du}=2f\frac{df}{du}$$

$$\Rightarrow 2\vec{f}\cdot\frac{d\vec{f}}{du}=2f\frac{df}{du}$$

$$\Rightarrow \vec{f}\cdot\frac{d\vec{f}}{du}=f\frac{df}{du}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

Necessity part :

If f is constant, then $\frac{df}{du} = 0$.

Then (1) gives $\vec{f} \cdot \frac{d\vec{f}}{du} = 0$.

Sufficiency part : If $\vec{f} \cdot \frac{d\vec{f}}{du} = 0, then\ by\ (1), f\frac{df}{du} = 0$.

Since $f \neq 0, \frac{df}{du} = 0\ and\ consequently\ f$ is a constant.

6. Show that the necessary and sufficient condition for the non-zero vector function $\vec{f}(u)$ to have a constant direction is $\vec{f} \times \frac{d\vec{f}}{du} = 0$. (i.e. $\vec{f}\ and \frac{d\vec{f}}{du}$ are parallel to each other).

Solution : Let $\vec{f} = f\hat{f}$, where f is the magnitude of $\vec{f}$.

Differentiating on both sides w.r.t. u, we get

$$\frac{d\vec{f}}{du} = \frac{df}{du}\hat{f} + \frac{fd\hat{f}}{du}.$$

$$\vec{f} \times \frac{d\vec{f}}{du} = \vec{f} \times \left\{\frac{df}{du}\hat{f} + f\frac{d\hat{f}}{du}\right\}$$

$$= \vec{f} \times \frac{df}{du}\hat{f} + \vec{f} \times f\frac{d\hat{f}}{du}$$

$$= \frac{df}{du}(\vec{f} \times \hat{f}\,) + f\left(\vec{f} \times \frac{d\hat{f}}{du}\right)$$

$$= \frac{df}{du}(\hat{f} \times f\hat{f}) + f\left(f\hat{f} \times \frac{d\hat{f}}{du}\right)$$

$$= 0 + f^2\left(\hat{f} \times \frac{d\hat{f}}{du}\right)$$

$$= f^2\left(\hat{f} \times \frac{d\hat{f}}{du}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

Necessity part : If $\vec{f}$ has a constant direction, then $\hat{f}$ is a constant vector and $\frac{d\hat{f}}{du} = 0$.

(1) implies that $\vec{f} \times \frac{d\vec{f}}{du} = 0$.

Sufficiency part : If $\vec{f} \times \frac{d\vec{f}}{du} = 0, then\ \hat{f} \times \frac{d\hat{f}}{du} = 0$ by (1).

So, eithr $\hat{f}\ and\ \frac{d\hat{f}}{du}$ are parallel or $\frac{d\hat{f}}{du}$ is a zero-vector.

Since they are not parallel but they are perpendicular. $\frac{d\hat{f}}{du} = 0$.

Hence the direction of $\hat{f}$ is constant, i.e., the direction of $\vec{f}$ is constant.

7. Show that, if $\vec{f}$ is not of constant direction, then $\left|\frac{d\vec{f}}{du}\right| \neq \frac{d}{du}|\vec{f}|$.

Solution : We have, $f^2 = \vec{f} \cdot \vec{f}$.

Differentiating w.r.t. u, we get $2f\frac{df}{du} = 2\vec{f} \cdot \frac{d\vec{f}}{du}$

$\Rightarrow f\frac{df}{du} = |\vec{f}|\ \left|\frac{d\vec{f}}{du}\right| cos\theta = f\left|\frac{d\vec{f}}{du}\right| cos\theta$,where θ is the angle between $\vec{f}$ and $\frac{d\vec{f}}{du}$.

$\Rightarrow \frac{df}{du} = \left|\frac{d\vec{f}}{du}\right| cos\theta.$

Since $\vec{f}$ is not of constant direction, $\vec{f}$ and $\frac{d\vec{f}}{du}$ are not parallel. Therefore, θ is not zero.

Hence, $\frac{df}{du} \neq \left|\frac{d\vec{f}}{du}\right|\ or\ \frac{d}{du}|\vec{f}| \neq \left|\frac{d\vec{f}}{du}\right|$.

8. Show that the necessary and sufficient condition for a vector function $\vec{f}(u)$ may be constant is $\frac{d\vec{f}}{du} = 0$.

Solution : **Necessity part :** Let $\vec{f}(u)$ be a constant vector.

Then $\frac{d\vec{f}}{du} = \lim_{u\to 0}\frac{\Delta\vec{f}}{\Delta u} = \lim_{\Delta u\to 0}\frac{\vec{f}\,(u+\Delta u)-\vec{f}\,(u)}{\Delta u}$

$$= \lim_{\Delta u\to 0}\frac{\vec{f}\,(u)-\vec{f}\,(u)}{\Delta u} = 0.$$

Sufficiency part:

Let $\vec{f} = f_1\vec{\imath} + f_2\vec{\jmath} + f_3\vec{k}$.

If $\frac{d\vec{f}}{du} = 0, then\ \frac{df_1}{du}\vec{\imath} + \frac{df_2}{du}\vec{\jmath} + \frac{df_3}{du}\vec{k} = 0.$

So, $\frac{df_1}{du} = 0$, $\frac{df_2}{du} = 0$, $\frac{df_3}{du}\vec{k} = 0$.

Thus, $f_1,\ f_2,\ f_3$ are constants and consequently $\vec{f}$ is a constant vector.

9. If $\vec{f}$ and $\vec{g}$ are vector functions of u such that $\vec{f} \times \frac{d\vec{g}}{du} = \vec{g} \times \frac{d\vec{f}}{du}$ for all values of u, show that $\vec{f}$ *and* $\vec{g}$ are always perpendicular to a fixed direction.

Solution : $\frac{d(\vec{f}\times\vec{g})}{du} = \vec{f} \times \frac{d\vec{g}}{du} + \frac{d\vec{f}}{du} \times \vec{g}$

$$= \vec{f} \times \frac{d\vec{g}}{du} - \vec{g} \times \frac{d\vec{f}}{du} = \vec{0}.$$

$\therefore$ $\vec{f} \times \vec{g}$ is a constant vector and hence in a fixed direction.

We know that $\vec{f} \times \vec{g}$ is perpendicular to both $\vec{f}$ *and* $\vec{g}$.

Therefore, $\vec{f}$ *and* $\vec{g}$ are perpendicular to a fixed directin.

10. If $\vec{e}$ is a variable unit vector depending on u, show that $\left|\vec{e} \times \frac{d\vec{e}}{du}\right| = \left|\frac{d\vec{e}}{du}\right|$.

Solution : Given $\vec{e}$ is a unit vector and so $\vec{e} \cdot \vec{e} = 1$.

Differentiating this with respect t u,

$$\frac{d\vec{e}}{du} \cdot \vec{e} + \vec{e} \cdot \frac{d\vec{e}}{du} = 0$$

$$\Rightarrow 2\left(\vec{e} \cdot \frac{d\vec{e}}{du}\right) = 0 \Rightarrow \vec{e} \cdot \frac{d\vec{e}}{du} = 0$$

$\therefore$ The angle between $\vec{e}$ and $\frac{d\vec{e}}{du}$ is 90°.

Now, $\vec{e} \times \frac{d\vec{e}}{du} = |\vec{e}| \left|\frac{d\vec{e}}{du}\right| sin90^\circ \hat{n} = \left|\frac{d\vec{e}}{du}\right| \hat{n}$ where $\hat{n}$ is a unit vector perpendicular to both $\vec{e}$ *and* $\frac{d\vec{e}}{du}$.

$$\therefore \left|\vec{e} \times \frac{d\vec{e}}{du}\right| = \left|\frac{d\vec{e}}{du}\right| \hat{n} = \left|\frac{d\vec{e}}{du}\right| \quad [\because |\hat{n}| = 1].$$

11. Show that, if $\vec{f}$ is a function of u, then

(i) $\frac{d}{du}[\vec{f}\ ,\vec{f}',\vec{f}''] = [\vec{f},\vec{f}',\vec{f}''']$

(ii) $\frac{d^2}{du^2}[\vec{f},\vec{f}',\vec{f}''] = [\vec{f},\vec{f}'',\vec{f}'''] + [\vec{f},\vec{f}',\vec{f}^{iv}]$, **where the dash denotes differentiation with respect to u.**

Solution : We know that $\frac{d}{du}[\vec{A},\vec{B},\vec{C}] = [\vec{A}',\vec{B},\vec{C}] + [\vec{A},\vec{B}',\vec{C}] + [\vec{A},\vec{B},\vec{C}']$.

(i) $\frac{d}{du}[\vec{f}\ ,\vec{f}',\vec{f}''] = [\vec{f}'\ ,\vec{f}',\vec{f}''] + [\vec{f}\ ,\vec{f}'',\vec{f}''] + [\vec{f}\ ,\vec{f}',\vec{f}''']$

$$= 0 + 0 + [\vec{f}\ ,\vec{f}',\vec{f}''']$$

$$= [\vec{f}\ ,\vec{f}',\vec{f}''']$$

(ii) $\frac{d^2}{du^2}[\vec{f},\vec{f}',\vec{f}''] = \frac{d}{du}\left(\frac{d}{du}[\vec{f},\vec{f}',\vec{f}'']\right)$

$$= \frac{d}{du}[\vec{f}\ ,\vec{f}',\vec{f}'''] \quad [from\ (i)]$$

$$= [\vec{f}'\ ,\vec{f}',\vec{f}'''] + [\vec{f}\ ,\vec{f}'',\vec{f}'''] + [\vec{f}\ ,\vec{f}',\vec{f}^{iv}]$$

$$= 0 + [\vec{f}\ ,\vec{f}'',\vec{f}'''] + [\vec{f}\ ,\vec{f}',\vec{f}^{iv}]$$

$$= [\vec{f}\ ,\vec{f}'',\vec{f}'''] + [\vec{f}\ ,\vec{f}',\vec{f}^{iv}].$$

12. If $[\vec{f},\vec{f}',\vec{f}''] = 0$, show that $\vec{f} \times \vec{f}'$ has a fixed direction and that $\vec{f}$ is parallel to a fixed plane.

Solution : Consider $(\vec{f} \times \vec{f}') \times \frac{d}{du}(\vec{f} \times \vec{f}') = (\vec{f} \times \vec{f}') \times [\vec{f} \times \vec{f}'' + \vec{f}' \times \vec{f}']$

$$= (\vec{f} \times \vec{f}') \times [\vec{f} \times \vec{f}'' + 0]$$

$$= (\vec{f} \times \vec{f}') \times [\vec{f} \times \vec{f}'']$$

$$= [\vec{f},\vec{f},\vec{f}'']\vec{f}' - [\vec{f}',\vec{f}\ ,\vec{f}'']\,\vec{f}$$

$$= 0 - [\vec{f}',\vec{f}\ ,\vec{f}'']\,\vec{f}$$

$$= [\vec{f}',\vec{f}\ ,\vec{f}'']\,\vec{f}$$

$$= 0.$$

$\therefore By$ problem 6, $\vec{f} \times \vec{f}'$ is a constant vector and $\frac{d}{du}(\vec{f} \times \vec{f}') = 0$.

$\therefore \vec{f} \times \vec{f}'$ has a fixed direction.

Also $\vec{f}$ is perpendicular to $\vec{f} \times \vec{f}'$ implies that $\vec{f}$ is parallel to a fixed plane.

13. If $\vec{r} = \vec{a}\, cos\omega t + \vec{b} sin\omega t$, where $\vec{a}, \vec{b}$ are constant vectors and ω, a constant scalar, show that $\vec{r} \times \frac{d\vec{r}}{dt} = \omega \vec{a} \times \vec{b}$, $\frac{d^2\vec{r}}{dt^2} = -\omega^2 \vec{r}$.

Solution : Given, $\vec{r} = \vec{a}\, cos\omega t + \vec{b} sin\omega t$.

Hence, $\frac{d\vec{r}}{dt} = -\omega\vec{a}\, sin\omega t + \omega\vec{b}\, cos\omega t$

$$= \omega(-\vec{a}\, sin\omega t + \vec{b}\, cos\omega t)$$

Now, $\vec{r} \times \frac{d\vec{r}}{dt} = (\vec{a}\, cos\omega t + \vec{b} sin\omega t) \times \omega(-\vec{a}\, sin\omega t + \vec{b}\, cos\omega t)$

$$= \omega(-cos\omega t\, sin\omega t\, \vec{a} \times \vec{a} + cos^2\omega t\, \vec{a} \times \vec{b} - sin^2\omega t\, \vec{b} \times \vec{a} + sin\omega t\, cos\omega t\, \vec{b} \times \vec{b})$$

$$= \omega(0 + cos^2\omega t\, \vec{a} \times \vec{b} + sin^2\omega t\, \vec{a} \times \vec{b} + 0)$$

$$= \omega(cos^2\omega t + sin^2\omega t)\left(\vec{a} \times \vec{b}\right)$$

$$= \omega\left(\vec{a} \times \vec{b}\right).$$

We have $\frac{d\vec{r}}{dt} = \omega(-\vec{a}\, sin\omega t + \vec{b}\, cos\omega t)$

$$\frac{d^2\vec{r}}{dt^2} = \omega(-\omega\vec{a}\, cos\omega t - \omega\vec{b}\, sin\omega t)$$

$$= -\omega^2(\vec{a}\, cos\omega t + \vec{b}\, sin\omega t)$$

$$= -\omega^2\vec{r}.$$

14. If $\vec{a}, \vec{b}, \vec{w}$ are vector functions of a scalar variable u and if $\frac{d\vec{a}}{du} = \vec{w} \times \vec{a}, \frac{d\vec{b}}{du} = \vec{w} \times \vec{b}$, then show that $\frac{d}{du}\left(\vec{a} \times \vec{b}\right) = \vec{w} \times \left(\vec{a} \times \vec{b}\right)$.

Solution : $\frac{d}{du}\left(\vec{a} \times \vec{b}\right) = \frac{d\vec{a}}{du} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{du}$

$$= (\vec{w} \times \vec{a}) \times \vec{b} + \vec{a} \times (\vec{w} \times \vec{b})$$

$$= (\vec{w} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{b})\vec{w} + (\vec{a} \cdot \vec{b})\vec{w} - (\vec{a} \cdot \vec{w})\vec{b}$$

$$= (\vec{w} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{w})\vec{b}$$

$$= \vec{w} \times (\vec{a} \times \vec{b}).$$

15. $\vec{t}, \vec{n}, \vec{b}$ are three mutually perpendicular unit vectors whose directions vary with a scalar variable u. Show that $\vec{t}', \vec{n}', \vec{b}'$ are coplanar, where the dash denotes differentiation with respect to u.

Solution : Given, $\vec{t}, \vec{n}, \vec{b}$ are three mutually perpendicular unit vectors.

$$\therefore\ \vec{t} = \vec{n} \times \vec{b} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

To prove, $\vec{t}', \vec{n}', \vec{b}'$ are coplanar. i.e., to prove $[\vec{t}', \vec{n}', \vec{b}'] = 0$.

Differentiating (1) w.r.t. u

$$\vec{t}' = \vec{n} \times \vec{b}' + \vec{n}' \times \vec{b}.$$

$$So, [\vec{t}', \vec{n}', \vec{b}'] = [\vec{n} \times \vec{b}' + \vec{n}' \times \vec{b}, \vec{n}', \vec{b}']$$

$$= [\vec{n} \times \vec{b}', \vec{n}', \vec{b}'] + [\vec{n}' \times \vec{b}, \vec{n}', \vec{b}']$$

$$= (\vec{n} \times \vec{b}') \cdot (\vec{n}' \times \vec{b}') + (\vec{n}' \times \vec{b}) \cdot (\vec{n}' \times \vec{b}')$$

$$= \begin{vmatrix} \vec{n} \cdot \vec{n}' & \vec{n} \cdot \vec{b}' \\ \vec{b}' \cdot \vec{n}' & \vec{b}' \cdot \vec{b}' \end{vmatrix} + \begin{vmatrix} \vec{n}' \cdot \vec{n}' & \vec{n}' \cdot \vec{b}' \\ \vec{b} \cdot \vec{n}' & \vec{b} \cdot \vec{b}' \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \vec{n} \cdot \vec{b}' \\ \vec{b}' \cdot \vec{n}' & \vec{b}' \cdot \vec{b}' \end{vmatrix} + \begin{vmatrix} \vec{n}' \cdot \vec{n}' & \vec{n}' \cdot \vec{b}' \\ \vec{b} \cdot \vec{n}' & 0 \end{vmatrix} \ [since, \vec{n} \cdot \vec{n}' = 0 = \vec{b} \cdot \vec{b}']$$

$$= -(\vec{n} \cdot \vec{b}')(\vec{b}' \cdot \vec{n}') - (\vec{n}' \cdot \vec{b}')(\vec{b} \cdot \vec{n}')$$

$$= -(\vec{b}' \cdot \vec{n}')[(\vec{n} \cdot \vec{b}') + (\vec{b} \cdot \vec{n}')]$$

$$= -(\vec{b}' \cdot \vec{n}')\frac{d}{du}(\vec{n} \cdot \vec{b})$$

$$= -(\vec{b}' \cdot \vec{n}')\frac{d}{du}(0)$$

$$= -(\vec{b}' \cdot \vec{n}')(0) = 0.$$

$\therefore \vec{t}', \vec{n}', \vec{b}'$ are coplanar.

16. The position vector of points on a curve are given by $\vec{r} = u^2\vec{\imath} - u\vec{\jmath} + (2u+1)\vec{k}$, where u is a parameter. Find the following at the point $u = 0$. $\frac{dr}{du}, \frac{d^2r}{du^2}, \left|\frac{dr}{du}\right|, \left|\frac{d^2r}{du^2}\right|$ and $\frac{dr}{ds} = \frac{\frac{dr}{du}}{\left|\frac{dr}{du}\right|}$.

[Ans. : $\frac{dr}{du} = -\vec{\jmath} + 2\vec{k}, \frac{d^2r}{du^2} = 2\vec{\imath}, \quad \left|\frac{dr}{du}\right| = \sqrt{5}, \quad \left|\frac{d^2r}{du^2}\right| = 2 \ and \frac{dr}{ds} = \frac{\frac{dr}{du}}{\left|\frac{dr}{du}\right|} = \frac{-\vec{\jmath}+2\vec{k}}{\sqrt{5}}$.

17. If $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ is a unit vector, show that $x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = 0$.

Solution : Given $\vec{r}$ is a unit vector.

$$\therefore \vec{r}\cdot\vec{r} = 1 \text{ and so } \vec{r}\cdot\frac{d\vec{r}}{dt} = 0.$$

$$\Rightarrow (x\vec{\imath} + y\vec{\jmath} + z\vec{k})\cdot\left(\frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}\right) = 0$$

$$\Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = 0.$$

1.5 Partial derivatives of vector function:

If for each pair of values of the scalar variables u and v there corresponds a vector $\vec{f}$, then $\vec{f}$ is said to be a vector function of u and v. Here the vector function is written specifically as $\vec{f}(u,v)$. Similarly, vector functions of several variables are defined.

Now we define partial derivatives of a vector function of two variables. Given the vector function $\vec{f}(u,v) of$ the variables u and v,

$$\lim_{\Delta u\to 0}\frac{\vec{f}(u+\Delta u,v)-\vec{f}(u,v)}{\Delta u} \text{ and } \lim_{\Delta v\to 0}\frac{\vec{f}(u,v+\Delta v)-\vec{f}(u,v)}{\Delta v}$$

are called the partial derivatives of $\vec{f}(u,v)$ with respect to u and v respectively. These partial derivatives are denoted, as in ordinary Calculus, by the symbols $\frac{\partial\vec{f}}{\partial u}, \frac{\partial\vec{f}}{\partial v}$ and the higher partial derivatives, by the symbol $\frac{\partial}{\partial u}\left(\frac{\partial\vec{f}}{\partial u}\right) = \frac{\partial^2\vec{f}}{\partial u^2}, \frac{\partial}{\partial u}\left(\frac{\partial\vec{f}}{\partial v}\right) = \frac{\partial^2\vec{f}}{\partial u\,\partial v}, etc.$

Similarly, the partial derivatives of vectors of more than two variables are defined.

Problem:

18. Prove the following results if $\vec{A}$ and $\vec{B}$ are vector functions of φ, a scalar function of the scalar parameters u and v:

(i) $\frac{\partial}{\partial u}(\vec{A}+\vec{B}) = \frac{\partial \vec{A}}{\partial u} + \frac{\partial \vec{B}}{\partial u}$

(ii) $\frac{\partial}{\partial u}(\varphi \vec{A}) = \left(\frac{\partial \varphi}{\partial u}\right)\vec{A} + \varphi\left(\frac{\partial \vec{A}}{\partial u}\right)$

(iii) $\frac{\partial}{\partial u}(\vec{A}\cdot\vec{B}) = \frac{\partial \vec{A}}{\partial u}\cdot\vec{B} + \vec{A}\cdot\frac{\partial \vec{B}}{\partial u}$

(iv) $\frac{\partial}{\partial u}(\vec{A}\times\vec{B}) = \frac{\partial \vec{A}}{\partial u}\times\vec{B} + \vec{A}\times\frac{\partial \vec{B}}{\partial u}$

Solution:

(i) $\frac{\partial}{\partial u}(\vec{A}+\vec{B}) = \lim_{\Delta u\to 0}\frac{\left(\vec{A}(u+\Delta u,v)+\vec{B}(u+\Delta u,v)\right)-\left(\vec{A}(u,v)+\vec{B}(u,v)\right)}{\Delta u}$

$$= \lim_{\Delta u\to 0}\frac{\left(\vec{A}(u+\Delta u,v)-\vec{A}(u,v)\right) + \left(\vec{B}(u+\Delta u,v)-\vec{B}(u,v)\right)}{\Delta u}$$

$$= \lim_{\Delta u\to 0}\frac{\left(\vec{A}(u+\Delta u,v)-\vec{A}(u,v)\right)}{\Delta u} + \lim_{\Delta u\to 0}\frac{\left((\vec{B}(u+\Delta u,v)-\vec{B}(u,v)\right)}{\Delta u}$$

$$= \frac{\partial \vec{A}}{\partial u} + \frac{\partial \vec{B}}{\partial u}.$$

(ii) $\frac{\partial}{\partial u}(\varphi\vec{A}) = \lim_{\Delta u\to 0}\frac{\left(\varphi(u+\Delta u,v)\vec{A}(u+\Delta u,v)\right)-\left(\varphi(u,v)\vec{A}(u,v)\right)}{\Delta u}$

$$= \lim_{\Delta u\to 0}\frac{\left(\varphi(u+\Delta u,v)\vec{A}(u+\Delta u,v)\right)-\varphi(u,v)\vec{A}(u+\Delta u,v)+\varphi(u,v)\vec{A}(u+\Delta u,v)-\left(\varphi(u,v)\vec{A}(u,v)\right)}{\Delta u}$$

$$= \lim_{\Delta u\to 0}\frac{[\varphi(u+\Delta u,v)-\varphi(u,v)]\vec{A}(u+\Delta u,v)}{\Delta u} + \lim_{\Delta u\to 0}\frac{\varphi(u,v)[\vec{A}(u+\Delta u,v)-\vec{A}(u,v)]}{\Delta u}$$

$$= \lim_{\Delta u\to 0}\frac{[\varphi(u+\Delta u,v)-\varphi(u,v)]}{\Delta u}\lim_{\Delta u\to 0}\vec{A}(u+\Delta u,v) +$$

$$\varphi(u,v)\lim_{\Delta u\to 0}\frac{[\vec{A}(u+\Delta u,v)-\vec{A}(u,v)]}{\Delta u}$$

$$= \left(\frac{\partial\varphi}{\partial u}\right)\vec{A}(u,v) + \varphi\left(\frac{\partial\vec{A}}{\partial u}\right)$$

$$= \left(\frac{\partial\varphi}{\partial u}\right)\vec{A} + \varphi\left(\frac{\partial\vec{A}}{\partial u}\right).$$

(iii) $\frac{\partial}{\partial u}(\vec{A}\cdot\vec{B}) = \lim_{\Delta u\to 0}\frac{\left(\vec{A}(u+\Delta u,v)\cdot\vec{B}(u+\Delta u,v)\right)-\left(\vec{A}(u,v)\cdot\vec{B}(u,v)\right)}{\Delta u}$

$$= \lim_{\Delta u \to 0} \frac{\left(\vec{A}(u+\Delta u, v) \cdot \vec{B}(u+\Delta u, v)\right) - \vec{A}(u,v) \cdot \vec{B}(u+\Delta u, v) + \vec{A}(u,v) \cdot \vec{B}(u+\Delta u, v) - \left(\vec{A}(u,v) \cdot \vec{B}(u,v)\right)}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{\left(\vec{A}(u+\Delta u, v) - \vec{A}(u,v)\right) \cdot \vec{B}(u+\Delta u, v)}{\Delta u} + \lim_{\Delta u \to 0} \frac{\vec{A}(u,v) \cdot \left(\vec{B}(u+\Delta u, v) - \vec{B}(u,v)\right)}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{\left(\vec{A}(u+\Delta u, v) - \vec{A}(u,v)\right)}{\Delta u} \cdot \lim_{\Delta u \to 0} \vec{B}(u+\Delta u, v) + \vec{A}(u,v) \cdot \lim_{\Delta u \to 0} \frac{\left(\vec{B}(u+\Delta u, v) - \vec{B}(u,v)\right)}{\Delta u}$$

$$= \frac{\partial \vec{A}}{\partial u} \cdot \vec{B} + \vec{A} \cdot \frac{\partial \vec{B}}{\partial u}.$$

(iv) $\frac{\partial}{\partial u}\left(\vec{A} \times \vec{B}\right) = \lim_{\Delta u \to 0} \frac{\left(\vec{A}(u+\Delta u, v) \times \vec{B}(u+\Delta u, v)\right) - \left(\vec{A}(u,v) \times \vec{B}(u,v)\right)}{\Delta u}$

$$= \lim_{\Delta u \to 0} \frac{\left(\vec{A}(u+\Delta u, v) \times \vec{B}(u+\Delta u, v)\right) - \vec{A}(u,v) \times \vec{B}(u+\Delta u, v) + \vec{A}(u,v) \times \vec{B}(u+\Delta u, v) - \left(\vec{A}(u,v) \times \vec{B}(u,v)\right)}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{\left(\vec{A}(u+\Delta u, v) - \vec{A}(u,v)\right) \times \vec{B}(u+\Delta u, v)}{\Delta u} + \lim_{\Delta u \to 0} \frac{\vec{A}(u,v) \times \left(\vec{B}(u+\Delta u, v) - \vec{B}(u,v)\right)}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{\left(\vec{A}(u+\Delta u, v) - \vec{A}(u,v)\right)}{\Delta u} \times \lim_{\Delta u \to 0} \vec{B}(u+\Delta u, v) + \vec{A}(u,v) \times \lim_{\Delta u \to 0} \frac{\left(\vec{B}(u+\Delta u, v) - \vec{B}(u,v)\right)}{\Delta u}$$

$$= \frac{\partial \vec{A}}{\partial u} \times \vec{B} + \vec{A} \times \frac{\partial \vec{B}}{\partial u}.$$

UNIT II

GRADIENT, DIVERGENCE AND CURL

2.1 Scalar point functions

If for every point P in a domain D of space, there corresponds a scalar $\emptyset$ then $\emptyset$ is said to be a single valued scalar point function defined in the domain D. The value of $\emptyset$ at P is denoted by $\emptyset(P)$ (or) $\emptyset(x,y,z) if\ P\ is\ (x,y,z)$. The function $\emptyset$ is said to be the scalar field in D.

Example : The temperature of a gas at different points in the region occupied by the gas define a scalar field in that region.

Vector point function

If for every point P in a domain D of space, there corresponds a vector $\emptyset$ then $\emptyset$ is said to be a single valued vector point function defined in the domain D. The value of $\emptyset$ at P is denoted by $\emptyset(P)$ (or) $\emptyset(x,y,z) if\ P\ is\ (x,y,z)$. The function $\emptyset$ is said to be the vector field in D.

Example : The velocity in a fluid motion is a vector point function.

Level surfaces

The surfaces represented by the equation $\emptyset = c$ for different values of c are called level surfaces. The values of $\emptyset$ at all points on a level surface are equal.

Result : No two level surfaces will intersect each other.

Directional derivative of a scalar point function

Theorem 2.1: The directional derivative of $\emptyset$ at any point P in the direction specified by the direction cosines l, m, n is $l\frac{\partial\emptyset}{\partial x} + m\frac{\partial\emptyset}{\partial y} + n\frac{\partial\emptyset}{\partial z}$.

Proof : Let P' be the point reached when one travels from $P(x,y,z)$ through a distance s in the given direction. Then P' has the co-ordinates $(x + ls, y + ms, z + ns)$.

The directional derivative of $\emptyset$ at P in the given direction is

$$\lim_{P' \to P} \frac{\emptyset(P') - \emptyset(P)}{PP'} = \lim_{s \to 0} \frac{\emptyset(x + ls, y + ms, z + ns) - \emptyset(x,y,z)}{s}.$$

By expanding $\emptyset(x+ls, y+ms, z+ns)$ in powers of s,

$$\emptyset(x+ls, y+ms, z+ns) = \emptyset(x,y,z) + \frac{1}{1!}\left(ls\frac{\partial}{\partial x} + ms\frac{\partial}{\partial y} + ns\frac{\partial}{\partial z}\right)\emptyset(x,y,z) +$$

$$1/2!\left(ls\frac{\partial}{\partial x} + ms\frac{\partial}{\partial y} + ns\frac{\partial}{\partial z}\right)^2 \emptyset(x,y,z) + \ldots\ldots\ldots\ldots$$

$$\therefore \frac{\emptyset(x+ls, y+ms, z+ns) - \emptyset(x,y,z)}{s} = \frac{1}{1!}\left(ls\frac{\partial}{\partial x} + ms\frac{\partial}{\partial y} + ns\frac{\partial}{\partial z}\right)\emptyset(x,y,z) +$$

$$1/2!\left(ls\frac{\partial}{\partial x} + ms\frac{\partial}{\partial y} + ns\frac{\partial}{\partial z}\right)^2 \emptyset(x,y,z) + \ldots\ldots\ldots\ldots$$

$$\lim_{P'\to P}\frac{\emptyset(P') - \emptyset(P)}{PP'} = \lim_{s\to 0}\frac{\emptyset(x+ls, y+ms, z+ns) - \emptyset(x,y,z)}{s}$$

$$= \lim_{s\to 0}\frac{\frac{1}{1!}\left(ls\frac{\partial}{\partial x} + ms\frac{\partial}{\partial y} + ns\frac{\partial}{\partial z}\right)\emptyset(x,y,z)}{s} +$$

$$\lim_{s\to 0}1/2!\left(ls\frac{\partial}{\partial x} + ms\frac{\partial}{\partial y} + ns\frac{\partial}{\partial z}\right)^2 \emptyset(x,y,z)/s + \ldots\ldots\ldots\ldots$$

$$= \left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y} + n\frac{\partial}{\partial z}\right)\emptyset(x,y,z) + 0 + 0 + \cdots\ldots$$

$$= l\frac{\partial\emptyset}{\partial x} + m\frac{\partial\emptyset}{\partial y} + n\frac{\partial\emptyset}{\partial z}.$$

2.2 Gradient of a scalar point function

Definition: If $\emptyset$ *is a* scalar point function, then the vector $\frac{\partial\emptyset}{\partial x}\vec{i} + \frac{\partial\emptyset}{\partial y}\vec{j} + \frac{\partial\emptyset}{\partial z}\vec{k}$ is called the gradient of $\emptyset$. This vector is written as grad$\emptyset$ or $\nabla\emptyset$ where ∇ (read as 'del' or 'nebla') stands for $\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$.

Note 2.1 The operator ∇ is an operator whose function is to transform a scalar point function $\emptyset$ into a vector point function.

Note 2.2 The summation notation for gradient is $\nabla\emptyset = \sum \vec{i}\frac{\partial\emptyset}{\partial x}$.

Theorem 2.2 : The directional derivative of $\emptyset$ in the direction specified by the unit vector $\vec{e}$ is $(\nabla\emptyset)\cdot\vec{e}$.

Proof: Let the direction cosines of $\vec{e}$ is l, m, n. Then $\vec{e} = l\vec{i} + m\vec{j} + n\vec{k}$.

Now, $\nabla\emptyset \cdot \vec{e} = \left(\frac{\partial\emptyset}{\partial x}\vec{i} + \frac{\partial\emptyset}{\partial y}\vec{j} + \frac{\partial\emptyset}{\partial z}\vec{k}\right) \cdot (l\vec{i} + m\vec{j} + n\vec{k}) = l\frac{\partial\emptyset}{\partial x} + m\frac{\partial\emptyset}{\partial y} + n\frac{\partial\emptyset}{\partial z}$

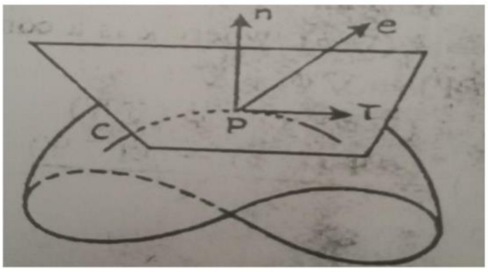
which is the directional derivative of $\emptyset$ in the direction whose direction cosines are l, m, n.

Note 2.3 : Maximum value of the directional derivative of $\emptyset$ is $|\nabla\emptyset|$.

Theorem 2.3 : (i) The direction of $\nabla\emptyset$ at P is normal to the level surface $\emptyset = c$ through P (i.e. $\nabla\emptyset$ is a vector along the normal at P to the level surface through P).

(ii) Magnitude of $\nabla\emptyset$ at P is the maximum of the directional derivative of $\emptyset$ at P.

Proof : **(i)** Suppose $\emptyset = c$ is the level surface through P and C is an arbitrarily chosen curve through P and on the level surface. Refer figure.



Now the tangent T to C at P lies in the tangent plane at P to the level surface. Since $\emptyset$ is a constant on C, the directional derivative of $\emptyset$ along C is zero. So, from diagram, if $\vec{T}$ is the unit vector along the tangent at P to C, then $(\nabla\emptyset) \cdot \vec{T} = 0$.

Hence $\nabla\emptyset$ at P is perpendicular to $\vec{T}$. But C is arbitrary. Therefore $\nabla\emptyset$ at P is perpendicular to the tangent plane to the level surface at P.

Hence $\nabla\emptyset$ at P is along the normal to the surface at P.

(ii) Let $\vec{n}$ be the unit vector along the normal to the level surface at P (in the sense in which $\emptyset$ increases).

Then $\nabla\emptyset = |\nabla\emptyset|\vec{n}$.

Directional derivative at P in an arbitrary direction $\vec{e} = (\nabla\emptyset) \cdot \vec{e} = |\nabla\emptyset|\vec{n} \cdot \vec{e} = |\nabla\emptyset|cos\theta$ where θ is the angle between $\vec{n}$ and $\vec{e}$.

$\therefore$ Maximum value of the directional derivative at $P = |\nabla\emptyset|$ since the maximum value of $cos\theta = 1$.

Hence the magnitude of $\nabla\emptyset$ is the maximum of the directional derivative.

Theorem 2.4 : If $\emptyset$ and ψ are scalar point functions, then prove that

(i) $\nabla(k\emptyset) = k(\nabla\emptyset)$ ***where k is a constant***

(ii) $\nabla(\emptyset + \psi) = \nabla\emptyset + \nabla\psi$

(iii) $\nabla(\emptyset\psi) = (\nabla\emptyset)\psi + \emptyset(\nabla\psi)$

(iv) $\nabla\left(\frac{\emptyset}{\psi}\right) = \frac{\psi(\nabla\emptyset) - \emptyset(\nabla\psi)}{\psi^2}$.

Proof : (i) $\nabla(k\emptyset) = \sum \vec{i}\frac{\partial(k\emptyset)}{\partial x} = \sum \vec{i}k\frac{\partial\emptyset}{\partial x} = k\sum\vec{i}\frac{\partial\emptyset}{\partial x} = k(\nabla\emptyset)$

(ii) $\nabla(\emptyset + \psi) = \sum\vec{i}\frac{\partial(\emptyset+\psi)}{\partial x} = \sum\vec{i}\frac{\partial\emptyset}{\partial x} + \sum\vec{i}\frac{\partial\psi}{\partial x} = \nabla\emptyset + \nabla\psi$

(iii) $\nabla(\emptyset\psi) = \sum\vec{i}\frac{\partial(\emptyset\psi)}{\partial x} = \sum\vec{i}(\frac{\partial\emptyset}{\partial x}\psi + \emptyset\frac{\partial\psi}{\partial x}) = \sum\vec{i}\frac{\partial\emptyset}{\partial x}\psi + \sum\vec{i}\emptyset\frac{\partial\psi}{\partial x}$

$=\psi\sum\vec{i}\frac{\partial\emptyset}{\partial x} + \emptyset\sum\vec{i}\frac{\partial\psi}{\partial x} = (\nabla\emptyset)\psi + \emptyset(\nabla\psi)$

(iv) $\nabla\left(\frac{\emptyset}{\psi}\right) = \sum\vec{i}\frac{\partial(\emptyset/\psi)}{\partial x} = \sum\vec{i}\left[\frac{\psi\frac{\partial\emptyset}{\partial x} - \emptyset\frac{\partial\psi}{\partial x}}{\psi^2}\right]$

$$= \frac{\sum\vec{i}\psi\frac{\partial\emptyset}{\partial x} + \sum\vec{i}\emptyset\frac{\partial\psi}{\partial x}}{\psi^2}$$

$$= \frac{\left(\psi\sum\vec{i}\frac{\partial\emptyset}{\partial x} + \emptyset\sum\vec{i}\frac{\partial\psi}{\partial x}\right)}{\psi^2}$$

$$= \frac{\psi(\nabla\emptyset) - \emptyset(\nabla\psi)}{\psi^2}$$

Problems

Problem 1: ***Find the directional derivative of $x + xy^2 + yz^3$ at the point $(0, 1, 1)$ in the direction whose d.c's are 2/3, 2/3, -1/3***

Soln : Let $\emptyset = x + xy^2 + yz^3$

Find $\frac{\partial\emptyset}{\partial x}, \frac{\partial\emptyset}{\partial y}, \frac{\partial\emptyset}{\partial z}$

Given $l = 2/3, m = 2/3 \text{ and } n = -1/3$

The directional derivative is $l\frac{\partial\emptyset}{\partial x} + m\frac{\partial\emptyset}{\partial y} + n\frac{\partial\emptyset}{\partial z}$

$=\frac{2}{3}(1+y^2)+\frac{2}{3}(2xy+z^3)-yz^2.$

At the point (0,1,1) [Ans. 1]

Problem 2. Find $\nabla\emptyset$ at (x,y,z) if $\emptyset = x + xy^2 + yz^3$

[Ans. :$\vec{i}(1+y^2)+\vec{j}(2xy+z^3)+\vec{k}(3yz^2)$]

Problem 3: Find the directional derivative of $\emptyset = x + xy^2 + yz^3$ at (0, 1, 1) in the direction of the vector $2\vec{i}+2\vec{j}-\vec{k}$. Find its maximum length.

Solution : Given, $\emptyset = x + xy^2 + yz^3$.

$$\vec{r} = 2\vec{i}+2\vec{j}-\vec{k}$$

$$|\vec{r}| = \sqrt{4+4+1} = 3$$

$$\therefore Unit\ vector\ \vec{e} = \frac{\vec{r}}{|\vec{r}|} = \frac{2\vec{i}+2\vec{j}-\vec{k}}{3} = \frac{2}{3}\vec{i}+\frac{2}{3}\ \vec{j}-\frac{1}{3}\vec{k}$$

$$\nabla\emptyset = \left(\frac{\partial\emptyset}{\partial x}\vec{i}+\frac{\partial\emptyset}{\partial y}\vec{j}+\frac{\partial\emptyset}{\partial z}\vec{k}\right) = \vec{i}(1+y^2)+\vec{j}(2xy+z^3)+\vec{k}(3yz^2)$$

$$\nabla\emptyset\cdot\vec{e} = [\vec{i}(1+y^2)+\vec{j}(2xy+z^3)+\vec{k}(3yz^2)]\cdot\left(\frac{2}{3}\vec{i}+\frac{2}{3}\ \vec{j}-\frac{1}{3}\vec{k}\right)$$

$$= \frac{2}{3}(1+y^2)+\frac{2}{3}(2xy+z^3)-\frac{1}{3}(3yz^2$$

$$\nabla\emptyset\cdot\vec{e}\ at\ (0,1,1) = \frac{2}{3}(2)+\frac{2}{3}(0+1)-1 = \frac{4}{3}+\frac{2}{3}-1 = 1.$$

Therefore, directional derivative of $\emptyset$ at (0,1,1) is 1.

Maximum length $=|\nabla\emptyset|$.

Now, $\nabla\emptyset = \vec{i}(1+1)+\vec{j}(2(0)+1)+\vec{k}(3) = 2\vec{i}+\vec{j}+3\vec{k}$

$$\therefore Maximum\ length = |\nabla\emptyset| = \sqrt{4+1+9} = \sqrt{14}.$$

Problem 4. Find the directional derivative of $\emptyset$ at the given point in the direction of the given vector.

(i) $\emptyset = 3xy^2 - x^2yz$ ***at the point (1,2,3) in the direction of the vector $\vec{i}-2\vec{j}+2\vec{k}$.***

{Hint : Find $\nabla\emptyset$ then find $\vec{e}$ as $\frac{1}{3}\vec{i}-\frac{2}{3}\vec{j}+\frac{2}{3}\vec{k}$. then find $\nabla\emptyset\cdot\vec{e}$. Ans. : -22/3}

(ii) $\emptyset = xyz - xy^2z^3$ *at* $(1,2,-1), \vec{r} = \vec{i} - \vec{j} - 3\vec{k}$. [Ans. $\frac{29}{\sqrt{11}}$]

(iii) $\emptyset = x^3 + y^3 + z^3$ ***at the point (1,-1,2) in the direction of the vector*** $\vec{i} + 2\vec{j} + \vec{k}$.

{Ans. $\frac{21}{\sqrt{6}}$}

Problem 5: ***If*** $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ ***(i.e.,) if*** $\vec{r}$ ***is the position vector of the variable point (x,y,z) and*** $|\vec{r}| = r$. ***Show that (i)*** $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$ ***and (ii)*** $\nabla\big(f(r)\big) = f'(r)\hat{r}$.

Proof : Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$ (i.e.,) $r^2 = x^2 + y^2 + z^2$

Differentiating partially with respect to x. $2r\frac{\partial r}{\partial x} = 2x$. $\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ *and* $\frac{\partial r}{\partial z} = \frac{z}{r}$.

Proof of (i) $\nabla\left(\frac{1}{r}\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right) = \vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \vec{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$$= \vec{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \vec{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + \vec{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= -\frac{1}{r^2}\vec{i}\left(\frac{\partial r}{\partial x}\right) - \frac{1}{r^2}\vec{j}\left(\frac{\partial r}{\partial y}\right) - \frac{1}{r^2}\vec{k}\left(\frac{\partial r}{\partial z}\right) = -\frac{1}{r^2}(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r})$$

$$= -\frac{1}{r^3}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = -\frac{\vec{r}}{r^3}$$

(ii) T.P. $\nabla\big(f(r)\big) = f'(r)\hat{r}$.

$$\nabla\big(f(r)\big) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(f(r))$$

$$= \vec{i}\frac{\partial}{\partial x}(f(r)) + \vec{j}\frac{\partial}{\partial y}(f(r)) + \vec{k}\frac{\partial}{\partial z}(f(r))$$

$$= f'(r)\vec{i}\left(\frac{\partial r}{\partial x}\right) + f'(r)\vec{j}\left(\frac{\partial r}{\partial y}\right) + f'(r)\vec{k}\left(\frac{\partial r}{\partial z}\right) = f'(r)(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r})$$

$$= \frac{f'(r)}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{f'(r)}{r}\vec{r} = f'(r)\hat{r} \quad \left[since, \frac{\vec{r}}{r} = \hat{r}\right].$$

Problem 6 : ***If*** $\nabla\emptyset = 5r^3\vec{r}$ ***then find*** $\emptyset$.

Solution : We have, $\frac{\vec{r}}{r} = \hat{r} \Rightarrow \vec{r} = r\hat{r}$

$$\therefore \nabla\emptyset = 5r^3 r\hat{r} = 5r^4\hat{r}$$

We have, $\nabla\emptyset = \emptyset'(r)\hat{r}$

$$\therefore \nabla\emptyset = \emptyset'(r)\hat{r} = 5r^4\hat{r}$$

$$\therefore \emptyset'(r) = 5r^4$$

Integrating with respect to r $\quad \int \emptyset'(r)dr = \int 5r^4 dr$

$\emptyset(r) = \frac{5r^5}{5} + c$ i.e., $\emptyset(r) = r^5 + c$.

Problem 7: If $\boldsymbol{\nabla\emptyset = (6r - 3r^2)\vec{r}}$ ***and*** $\boldsymbol{\emptyset(2) = 4}$ ***then find*** $\boldsymbol{\emptyset}$.

{Hint : Find the value of c using the condition $\emptyset(2) = 4$. (Ans. $\emptyset(r) = 2(r^3 - r^4 + 10)$.

Problem 8***: If*** $\boldsymbol{\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}}$ ***(i.e.,) if*** $\boldsymbol{\vec{r}}$ ***is the position vector of the variable point (x,y,z) and*** $\boldsymbol{|\vec{r}| = r}$***, then show that***

- ***(i)*** $\boldsymbol{\nabla(log r) = \frac{\vec{r}}{r^2}}$
- ***(ii)*** $\boldsymbol{\nabla r^n = nr^{n-1}\hat{r} = nr^{n-2}\vec{r}}$
- ***(iii)*** $\boldsymbol{\nabla(\vec{r}\cdot\vec{a}) = \vec{a}}$ ***where a is a constant vector.***
- ***(iv)*** $\boldsymbol{\nabla(\vec{a}\cdot\vec{r}) = 2a}$ ***if*** $\boldsymbol{\vec{a} = \alpha x\vec{i} + \beta y\vec{j} + \gamma z\vec{k}}$.
- ***(v)*** $\boldsymbol{\frac{dF}{dt} = \frac{\partial F}{\partial t} + (\nabla F)\cdot\frac{d\vec{r}}{dt}}$ ***where F is a scalar function of t and x, y, z are functions of t.***

Proof : Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$ (i.e.,) $r^2 = x^2 + y^2 + z^2$

Differentiating partially with respect to x. $2r\frac{\partial r}{\partial x} = 2x. \quad \therefore \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}\ and\ \frac{\partial r}{\partial z} = \frac{z}{r}$

(i) $\quad \nabla(log r) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(log r)$

$$= \vec{i}\frac{\partial}{\partial x}(log r) + \vec{j}\frac{\partial}{\partial y}(log r) + \vec{k}\frac{\partial}{\partial z}(log r)$$

$$= \vec{i}\left(\frac{1}{r}\frac{\partial r}{\partial x}\right) + \vec{j}\left(\frac{1}{r}\frac{\partial r}{\partial y}\right) + \vec{k}\left(\frac{1}{r}\frac{\partial r}{\partial z}\right)$$

$$= \frac{1}{r}\vec{i}\left(\frac{\partial r}{\partial x}\right) + \frac{1}{r}\vec{j}\left(\frac{\partial r}{\partial y}\right) + \frac{1}{r}\vec{k}\left(\frac{\partial r}{\partial z}\right) = \frac{1}{r}(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r})$$

$$= \frac{1}{r^2}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{\vec{r}}{r^2}$$

(ii) $\nabla r^n = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(r^n)$

$$= \vec{i}\frac{\partial(r^n)}{\partial x} + \vec{j}\frac{\partial(r^n)}{\partial y} + \vec{k}\frac{\partial(r^n)}{\partial z}$$

$$= \vec{i}nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right)$$

$$= nr^{n-1}\frac{\vec{r}}{r} = nr^{n-1}\frac{\vec{r}}{|\vec{r}|} = nr^{n-1}\hat{r}.$$

(iii) T.P. $\nabla(\vec{r}\cdot\vec{a}) = \vec{a}$

Given a is a constant vector. $\therefore \vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{r}\cdot\vec{a} = (x\vec{i} + y\vec{j} + z\vec{k})\cdot(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) = a_1x + a_2y + a_3z$

$$\nabla(\vec{r}\cdot\vec{a}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$

$$= \vec{i}\frac{\partial}{\partial x}(a_1x + a_2y + a_3z) + \vec{j}\frac{\partial}{\partial y}(a_1x + a_2y + a_3z) + \vec{k}\frac{\partial}{\partial z}(a_1x + a_2y + a_3z)$$

$$= a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = \vec{a}$$

(iv) Given $\vec{a} = \alpha x\vec{i} + \beta y\vec{j} + \gamma z\vec{k}$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{a}\cdot\vec{r} = (\alpha x\vec{i} + \beta y\vec{j} + \gamma z\vec{k})\cdot(x\vec{i} + y\vec{j} + z\vec{k}) = \alpha x^2 + \beta y^2 + \gamma z^2$ *where* α, β, γ are constants.

$$\nabla(\vec{a}\cdot\vec{r}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\alpha x^2 + \beta y^2 + \gamma z^2)$$

$$= \vec{i}\frac{\partial}{\partial x}(\alpha x^2 + \beta y^2 + \gamma z^2) + \vec{j}\frac{\partial}{\partial y}(\alpha x^2 + \beta y^2 + \gamma z^2) + \vec{k}\frac{\partial}{\partial z}(\alpha x^2 + \beta y^2 + \gamma z^2)$$

$$= 2\alpha x\vec{i} + 2\beta y\vec{j} + 2\gamma z\vec{k} = 2(\alpha x\vec{i} + \beta y\vec{j} + \gamma z\vec{k}) = 2\vec{a}.$$

(v) $\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt}$ since F is a function of t and x, y ,z are functions of t.

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \left(\frac{\partial F}{\partial x}\vec{\imath} + \frac{\partial F}{\partial y}\vec{\jmath} + \frac{\partial F}{\partial z}\vec{k}\right) \cdot \left(\frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}\right)$$

Thus

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + (\nabla F) \cdot \left(\frac{d\vec{r}}{dt}\right).$$

Problem 9: If $\boldsymbol{\nabla\emptyset = (y + y^2 + z^2)\vec{\imath} + (x + z + 2xy)\vec{\jmath} + (y + 2zx)\vec{k}}$ ***and if*** $\boldsymbol{\emptyset(1,1,1) = 3, find\ \emptyset.}$

Solution : Given, $\nabla\emptyset = (y + y^2 + z^2)\vec{\imath} + (x + z + 2xy)\vec{\jmath} + (y + 2zx)\vec{k}$ ..... (1)

We have, $\nabla\emptyset = \frac{\partial\emptyset}{\partial x}\vec{\imath} + \frac{\partial\emptyset}{\partial y}\vec{\jmath} + \frac{\partial\emptyset}{\partial z}\vec{k}$ ....(2)

From (1) and (2) we have $\frac{\partial\emptyset}{\partial x} = (y + y^2 + z^2)$ ......(3)

$\frac{\partial\emptyset}{\partial y} = (x + z + 2xy)$ .....(4) & $\frac{\partial\emptyset}{\partial z} = y + 2zx$ ......(5)

Integrating (3) w.r.to x, $\emptyset = yx + y^2x + xz^2 + f(y, z)$ .... (6)

Integrating (4) w.r. to y, $\emptyset = xy + zy + xy^2 + g(x, z)$ ...... (7)

Integrating (5) w.r.to z, $\emptyset = yz + z^2x + h(x, y)$....... (8)

From (6), (7) and (8) we get, $\emptyset = yx + y^2x + xz^2 + yz + c$

$Given, \emptyset(1,1,1) = 3$. Therefore, 1+1+1+1+c=3 => $c = -1$

Hence, $\emptyset = yx + y^2x + xz^2 + yz - 1.$

Problem 10: Find $\boldsymbol{\emptyset}$ ***if*** $\boldsymbol{\nabla\emptyset}$ ***is*** $\boldsymbol{(6xy + z^3)\vec{\imath} + (3x^2 - z)\vec{\jmath} + (3xz^2 - y)\vec{k}}$

(Ans. : $3x^2y + xz^3 - yz + c$)

Problem 11: Find the unit vectors normal to the following surfaces.

(i) $\boldsymbol{x^2 + 2y^2 + z^2 = 7\ at\ (1, -1, 2)}$

(ii) $\boldsymbol{x^2 + y^2 - z^2 = 1\ at\ (1, 1, 1)}$ ***[Ans.*** $\boldsymbol{\frac{\vec{\imath}+\vec{\jmath}-\vec{k}}{\sqrt{3}}}$***]***

(iii) $\boldsymbol{x^2 + 3y^2 + 2z^2 = 6\ at\ (2, 0, 1)}$ ***[Ans.*** $\boldsymbol{\frac{\vec{\imath}+\vec{k}}{\sqrt{2}}}$***]***

Solution : (i)Let $\emptyset = x^2 + 2y^2 + z^2 - 7$

$$\nabla\emptyset = 2x\vec{i} + 4y\vec{j} + 2z\vec{k}$$

At $(1,-1,2), \nabla\emptyset = 2\vec{i} - 4\vec{j} + 4\vec{k}$

$$|\nabla\emptyset| = 6.$$

Unit vector normal to the surface = $\frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{\vec{i}-2\vec{j}+2\vec{k}}{3}$

Problem 12 : Find the equation of the tangent plane to the surface $x^2 + 2y^2 + 3z^2 = 6$ at the point (1,-1,1).

Sol. : Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Let $\emptyset = x^2 + 2y^2 + 3z^2 - 6$

$$\nabla\emptyset = 2x\vec{i} + 4y\vec{j} + 6z\vec{k}$$

At $(1,-1,1), \nabla\emptyset = 2\vec{i} - 4\vec{j} + 6\vec{k} = \vec{P}$

Let $\vec{r_1} = \vec{i} - \vec{j} + \vec{k}$

Equation of the tangent plane is $(\vec{r} - \vec{r_1}) \cdot \vec{P} = 0$

$$\left(x\vec{i} + y\vec{j} + z\vec{k} - (\vec{i} - \vec{j} + \vec{k})\right) \cdot (2\vec{i} - 4\vec{j} + 6\vec{k}) = 0$$

$$i.e., x - 2y + 3z - 6 = 0 .$$

Problem 13 : Find the equation of the tangent plane to the surface $x^2 - 4y^2 + 3z^2 + 4 = 0$ at the point (3,2,1). [Ans. 3x-8y+3z+4=0]

Problem 14: Find the angle between the normals to the surface $xy - z^2 = 0$ at the points (1,4,-2) and (-3,-3,3)

Solution : First find $\nabla\emptyset$ at the points (1,4,-2) and (-3,-3,3).

At $(1,4,-2), \nabla\emptyset = 4\vec{i} + \vec{j} + 4\vec{k}$ & At $(-3,-3,3), \nabla\emptyset = -3\vec{i} - 3\vec{j} - 6\vec{k}$

If θ is the angle between the normals then $cos\theta = \frac{(4\vec{i}+\vec{j}+4\vec{k})\cdot(-3\vec{i}-3\vec{j}-6\vec{k})}{\sqrt{16+1+16}\sqrt{9+9+36}} = \frac{-13}{3\sqrt{22}}$

Problem 14: Show that the surfaces $5x^2 - 2yz - 9x = 0$ and $4x^2y + z^3 - 4 = 0$ are orthogonal at (1,-1,2).

Soln. : - Let $\emptyset_1 = 5x^2 - 2yz - 9x$ and $\emptyset_2 = 4x^2y + z^3 - 4$

$$\nabla\emptyset_1 = (10x-9)\vec{\imath} - 2z\vec{\jmath} - 2y\vec{k} \; \& \; \nabla\emptyset_2 = 8xy\vec{\imath} + 4x^2\vec{\jmath} + 3z^2\vec{k}$$

At (1,-1,2) $\nabla\emptyset_1 = \vec{\imath} - 4\vec{\jmath} + 2\vec{k} \; \& \; \nabla\emptyset_2 = -8\vec{\imath} + 4\vec{\jmath} + 12\vec{k}$

$$(\nabla\emptyset_1)\cdot(\nabla\emptyset_2) = (\vec{\imath} - 4\vec{\jmath} + 2\vec{k}\,)\cdot(-8\vec{\imath} + 4\vec{\jmath} + 12\vec{k}\,) = 0$$

Thus the given two surfaces are orthogonal.

Problem 15: Find the angle between the normals to the intersecting surfaces $xy - z^2 - 1 = 0$ and $y^2 - 3z - 1 = 0$ at (1,1,0). Also find a unit vector along the tangent to the curve of intersection of the surfaces at (1,1,0).

Soln. : As in the previous problem find $\nabla\emptyset_1 \; \& \; \nabla\emptyset_2 \; at \; (1,1,0)$

$$\nabla\emptyset_1 = \vec{\imath} + \vec{\jmath} \; \& \; \nabla\emptyset_2 = 2\vec{\jmath} - 3\vec{k}$$

Let $\vec{a} = \vec{\imath} + \vec{\jmath} \; \& \; \vec{b} = 2\vec{\jmath} - 3\vec{k}$

Let θ bet the angle between the normals to the surfaces.

$$\therefore cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{2}{\sqrt{26}}$$

Unit Vector along the tangent = $\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 1 & 0 \\ 0 & 2 & -3 \end{vmatrix} = -3\vec{\imath} + 3\vec{\jmath} + 2\vec{k} \text{ and } |\vec{a}\times\vec{b}| = \sqrt{22} \text{ (Verify)}$$

Thus the unit vector along the tangent = $\frac{-3\vec{\imath}+3\vec{\jmath}+2\vec{k}}{\sqrt{22}}$.

Problem 16: Find the direction in which $\emptyset = xy^2 + yz^2 + zx^2$ increases most rapidly at the point (1,2,-3).

Soln. : Find $\nabla\emptyset$ at (1,2,-3)

Direction of $\nabla(xy^2 + yz^2 + zx^2) = -2\vec{\imath} + 13\vec{\jmath} - 11\vec{k}$.

2.3 Divergence and curl of a vector point function

Definition: If $V = V_1\vec{\imath} + V_2\vec{\jmath} + V_3\vec{k}$ is a vector point function, then the scalar $\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$ is called the **divergence of V** and is denoted by divV (or) $\nabla\cdot V$.

If $\nabla \cdot V = 0$ then the vector V is said to be **solenoidal**.

The summation notation for divergence is $\nabla \cdot V = \sum \vec{\imath} \cdot \frac{\partial V}{\partial x}$.

Definition : If $\boldsymbol{V} = V_1\vec{\imath} + V_2\vec{\jmath} + V_3\vec{k}$, then the vector $\vec{\imath}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \vec{\jmath}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \vec{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$ is called the **curl of V** and is denoted by **curlV** (or) $\nabla \times V$.

$$\text{Now, } \nabla \times V = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

If $\boldsymbol{\nabla \times V = 0}$ then the vector **V** is said to be **irrotational**.

Note 2.4 : The divergence of a vector point function is a scalar and the curl of a vector point function is a vector.

Note 2.5 : $V \cdot \nabla = V \cdot \sum \vec{\imath} \frac{\partial}{\partial x}$

Theorem 2.5 : If $\vec{A}$ and $\vec{B}$ are vector point functions, '∅' a scalar point function and 'k' a constant then, (i) $\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$

(ii) $\nabla \cdot (k\vec{A}) = k(\nabla \cdot \vec{A})$

(iii) $\nabla \cdot (\emptyset\vec{A}) = (\nabla\emptyset) \cdot \vec{A} + \emptyset(\nabla \cdot \vec{A})$

(iv) $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$

(v) $\nabla \times (k\vec{A}) = k(\nabla \times \vec{A})$

(vi) $\nabla \times (\emptyset\vec{A}) = (\nabla\emptyset) \times \vec{A} + \emptyset(\nabla \times \vec{A})$

Proof : (i) $\nabla \cdot (\vec{A} + \vec{B}) = \sum \vec{\imath} \cdot \frac{\partial(\vec{A}+\vec{B})}{\partial x} = \sum \vec{\imath} \cdot (\frac{\partial \vec{A}}{\partial x} + \frac{\partial \vec{B}}{\partial x}) = \sum \vec{\imath} \cdot \frac{\partial \vec{A}}{\partial x} + \sum \vec{\imath} \cdot \frac{\partial \vec{B}}{\partial x}$

$= \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$

(ii) $\nabla \cdot (k\vec{A}) = \sum \vec{\imath} \cdot \frac{\partial(k\vec{A})}{\partial x} = k \sum \vec{\imath} \cdot \frac{\partial \vec{A}}{\partial x}$ [Since, k is a constant]$= k(\nabla \cdot \vec{A})$

(iii) $\nabla \cdot (\emptyset\vec{A}) = \sum \vec{\imath} \cdot \frac{\partial(\emptyset\vec{A})}{\partial x} = \sum \vec{\imath} \cdot [\frac{\partial \emptyset}{\partial x}\vec{A} + \emptyset\frac{\partial \vec{A}}{\partial x}] = \sum \vec{\imath} \cdot \frac{\partial \emptyset}{\partial x}\vec{A} + \sum \vec{\imath} \cdot \emptyset\frac{\partial \vec{A}}{\partial x}$

$= \sum \vec{\imath}\frac{\partial \emptyset}{\partial x} \cdot \vec{A} + \emptyset \sum \vec{\imath} \cdot \frac{\partial \vec{A}}{\partial x} = (\nabla\emptyset) \cdot \vec{A} + \emptyset(\nabla \cdot \vec{A})$

(iv) $\nabla \times (\vec{A} + \vec{B}) = \sum \vec{i} \times \frac{\partial(\vec{A}+\vec{B})}{\partial x} = \sum \vec{i} \times (\frac{\partial \vec{A}}{\partial x} + \frac{\partial \vec{B}}{\partial x}) = \sum \vec{i} \times \frac{\partial \vec{A}}{\partial x} + \sum \vec{i} \times \frac{\partial \vec{B}}{\partial x}$

$$= \nabla \times \vec{A} + \nabla \times \vec{B}$$

(v) $\nabla \times (k\vec{A}) = \sum \vec{i} \times \frac{\partial(k\vec{A})}{\partial x} = k \sum \vec{i} \times \frac{\partial \vec{A}}{\partial x}$ [Since, k is a constant]$= k(\nabla \times \vec{A})$

(vi) $\nabla \times (\emptyset\vec{A}) = \sum \vec{i} \times \frac{\partial(\emptyset\vec{A})}{\partial x} = \sum \vec{i} \times [\frac{\partial \emptyset}{\partial x}\vec{A} + \emptyset\frac{\partial \vec{A}}{\partial x}] = \sum \vec{i} \times \frac{\partial \emptyset}{\partial x}\vec{A} + \sum \vec{i} \times \emptyset\frac{\partial \vec{A}}{\partial x}$

$$= \sum \vec{i}\frac{\partial \emptyset}{\partial x} \times \vec{A} + \emptyset \sum \vec{i} \times \frac{\partial \vec{A}}{\partial x} = (\nabla\emptyset) \times \vec{A} + \emptyset(\nabla \times \vec{A})$$

Theorem 2.6 : If $\vec{A}$ and $\vec{B}$ are vector point functions then,

(i) $\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + (\vec{A} \cdot \nabla)\vec{B} + \vec{B} \times (\nabla \times \vec{A}) + (\vec{B} \cdot \nabla)\vec{A}$

(ii) $\nabla \cdot (\vec{A} \times \vec{B}) = (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A}$

(iii) $\nabla \times (\vec{A} \times \vec{B}) = \{(\vec{B} \cdot \nabla)\vec{A} - (\nabla \cdot \vec{A})\vec{B}\} - \{(\vec{A} \cdot \nabla)\vec{B} - (\nabla \cdot \vec{B})\vec{A}\}$

Proof : (i) $\nabla(\vec{A} \cdot \vec{B}) = \sum \vec{i} \cdot \frac{\partial(\vec{A}\cdot\vec{B})}{\partial x}$

$$= \sum \vec{i}\left[\frac{\partial \vec{A}}{\partial x} \cdot \vec{B} + \vec{A} \cdot \frac{\partial \vec{B}}{\partial x}\right]$$

$$= \sum \vec{i}\left[\frac{\partial \vec{A}}{\partial x} \cdot \vec{B}\right] + \sum \vec{i}\left[\vec{A} \cdot \frac{\partial \vec{B}}{\partial x}\right]$$

$$= \sum \vec{i}\left[\vec{B} \cdot \frac{\partial \vec{A}}{\partial x}\right] + \sum \vec{i}\left[\vec{A} \cdot \frac{\partial \vec{B}}{\partial x}\right] \text{.............. (1)}$$

Now, $\vec{A} \times \left(\vec{i} \times \frac{\partial \vec{B}}{\partial x}\right) = \left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{i} - (\vec{A} \cdot \vec{i})\frac{\partial \vec{B}}{\partial x}$

(Using, $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$)

$$\vec{A} \times \left(\vec{i} \times \frac{\partial \vec{B}}{\partial x}\right) + (\vec{A} \cdot \vec{i})\frac{\partial \vec{B}}{\partial x} = \left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{i}$$

Taking summation on both sides.

$$\sum[\vec{A} \times \left(\vec{i} \times \frac{\partial \vec{B}}{\partial x}\right) + (\vec{A} \cdot \vec{i})\frac{\partial \vec{B}}{\partial x}] = \sum[\left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{i}]$$

$$\sum[\vec{A} \times \left(\vec{i} \times \frac{\partial \vec{B}}{\partial x}\right)] + \sum[(\vec{A} \cdot \vec{i})\frac{\partial \vec{B}}{\partial x}] = \sum[\left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{i}]$$

$$\sum\left[\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{\imath}\right] = \sum\left[\vec{A}\times\left(\vec{\imath}\times\frac{\partial\vec{B}}{\partial x}\right)\right] + \sum\left[(\vec{A}\cdot\vec{\imath})\frac{\partial\vec{B}}{\partial x}\right]$$

$$= \vec{A}\times\sum\left(\vec{\imath}\times\frac{\partial\vec{B}}{\partial x}\right) + \left[\vec{A}\cdot\sum\vec{\imath}\frac{\partial}{\partial x}\right]\vec{B}$$

$$= \vec{A}\times(\nabla\times\vec{B})+(\vec{A}\cdot\nabla)\vec{B} \quad \text{......................} \ (2)$$

Interchanging $\vec{A}$ and $\vec{B}$, we get

$$\sum\left[\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{\imath}\right] = \vec{B}\times(\nabla\times\vec{A})+(\vec{B}\cdot\nabla)\vec{A} \quad \text{........................} \ (3)$$

Put (2) and (3) in (1) we get,

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + (\vec{B}\cdot\nabla)\vec{A}$$

(ii) $\nabla\cdot(\vec{A}\times\vec{B}) = \sum\vec{\imath}\cdot\frac{\partial(\vec{A}\times\vec{B})}{\partial x} = \sum\vec{\imath}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B} + \vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)$

$$= \sum\vec{\imath}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right) + \sum\vec{\imath}\cdot\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right) = \sum\vec{\imath}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right) - \sum\vec{\imath}\cdot\left(\frac{\partial\vec{B}}{\partial x}\times\vec{A}\right)$$

$$= \sum\vec{\imath}\times\left(\frac{\partial\vec{A}}{\partial x}\cdot\vec{B}\right) - \sum\vec{\imath}\times\left(\frac{\partial\vec{B}}{\partial x}\cdot\vec{A}\right) \quad \text{[Interchanging dot and cross]}$$

$$= \sum\left(\vec{\imath}\times\frac{\partial\vec{A}}{\partial x}\right)\cdot\vec{B} - \sum\left(\vec{\imath}\times\frac{\partial\vec{B}}{\partial x}\right)\cdot\vec{A} = (\nabla\times\vec{A})\cdot\vec{B} - (\nabla\times\vec{B})\cdot\vec{A}$$

(iii) $\nabla\times(\vec{A}\times\vec{B}) = \sum\vec{\imath}\times\frac{\partial(\vec{A}\times\vec{B})}{\partial x} = \sum\vec{\imath}\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B} + \vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)$

$$= \sum\vec{\imath}\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right) + \sum\vec{\imath}\times\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)$$

$$= \sum\vec{\imath}\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right) - \sum\vec{\imath}\times\left(\frac{\partial\vec{B}}{\partial x}\times\vec{A}\right) \quad \text{......................} \ (1)$$

Now, $\vec{\imath}\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right) = (\vec{\imath}\cdot\vec{B})\frac{\partial\vec{A}}{\partial x} - \left(\vec{\imath}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{B}$

$$\sum\vec{\imath}\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right) = \sum(\vec{\imath}\cdot\vec{B})\frac{\partial\vec{A}}{\partial x} - \sum\left(\vec{\imath}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{B}$$

$$= \sum(\vec{B}\cdot\vec{\imath})\frac{\partial\vec{A}}{\partial x} - \sum\left(\vec{\imath}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{B} = \vec{B}\cdot\sum\vec{\imath}\frac{\partial\vec{A}}{\partial x} - \sum\left(\vec{\imath}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{B}$$

$$= \left(\vec{B}\cdot\sum\vec{\imath}\frac{\partial}{\partial x}\right)\vec{A} - \sum\left(\vec{\imath}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{B}$$

$= (\vec{B}\cdot\nabla)\vec{A} - (\nabla\cdot\vec{A})\vec{B}$ .... (2)

Interchanging $\vec{A}$ & $\vec{B}$ we get, $\sum \vec{\iota}\times\left(\frac{\partial\vec{B}}{\partial x}\times\vec{A}\right) = (\vec{A}\cdot\nabla)\vec{B} - (\nabla\cdot\vec{B})\vec{A}$ .... (3)

Put (2) and (3) in (1) we get the result.

Problem 17 : Show that the vector $\vec{A} = x^2z^2\vec{\iota} + xyz^2\vec{\jmath} - xz^3\vec{k}$ is solenoidal.

Solution : $\nabla\cdot\vec{A} = \left(\vec{\iota}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(x^2z^2\vec{\iota} + xyz^2\vec{\jmath} - xz^3\vec{k}\right)$

i.e., $\frac{\partial(x^2z^2)}{\partial x} + \frac{\partial(xyz^2)}{\partial y} + \frac{\partial(-xz^3)}{\partial z} = 2xz^2 + xz^2 - 3xz^2 = 3xz^2 - 3xz^2 = 0.$

Problem 18: If the vector $3x\vec{\iota} + (x+y)\vec{\jmath} - az\vec{k}$ is solenoidal, find a.

Solution : Let $\vec{A} = 3x\vec{\iota} + (x+y)\vec{\jmath} - az\vec{k}$

Given $\vec{A}$ is solenoidal. Therefore, $\nabla\cdot\vec{A}=0$

i.e., $\left(\vec{\iota}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(3x\vec{\iota} + (x+y)\vec{\jmath} - az\vec{k}\right) = 0$

i.e., $\frac{\partial(3x)}{\partial x} + \frac{\partial(x+y)}{\partial y} - \frac{\partial(az)}{\partial z} = 0$

i.e., 3+1-a=0 =>a=4.

Problem 19 : If $\vec{r} = x\vec{\iota} + y\vec{\jmath} + z\vec{k}$, Show that $\nabla\cdot\vec{r} = 3$.

Solution : Given $\vec{r} = x\vec{\iota} + y\vec{\jmath} + z\vec{k}$

$$\nabla\cdot\vec{r} = \left(\vec{\iota}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(x\vec{\iota} + y\vec{\jmath} + z\vec{k}\right)$$

$$= \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z} = 1 + 1 + 1 = 3$$

Problem 20 : If $\vec{r} = x\vec{\iota} + y\vec{\jmath} + z\vec{k}$ and $r = |\vec{r}|$ show that $\nabla\cdot(r^n\vec{r}) = (n+3)r^n$.

Solution : $\nabla\cdot(r^n\vec{r}) = \nabla(r^n)\cdot\vec{r} + r^n(\nabla\cdot\vec{r})$ .... (1)

$$\nabla(r^n) = \vec{\iota}\frac{\partial(r^n)}{\partial x} + \vec{\jmath}\frac{\partial(r^n)}{\partial y} + \vec{k}\frac{\partial(r^n)}{\partial z}$$

$$= \vec{\iota}nr^{n-1}\frac{\partial r}{\partial x} + \vec{\jmath}nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}nr^{n-1}\frac{\partial r}{\partial z}$$

$$= \vec{i}nr^{n-1}\frac{x}{r} + \vec{j}nr^{n-1}\frac{y}{r} + \vec{k}nr^{n-1}\frac{z}{r}$$

$$= nr^{n-2}(\vec{i}x + \vec{j}y + \vec{k}z) = nr^{n-2}\vec{r}$$

$$\therefore (1) \Rightarrow \nabla \cdot (r^n\vec{r}) = (nr^{n-2}\vec{r} \cdot \vec{r}) + r^n 3 = nr^{n-2}(\vec{r} \cdot \vec{r}) + r^n 3$$

$$= nr^{n-2}r^2 + r^n 3 = nr^n + r^n 3 = (n+3)r^n$$

Problem 21 : Show that $\boldsymbol{\nabla \cdot \left(\frac{1}{r^3}\vec{r}\right) = 0}$ & $\nabla \cdot \hat{\boldsymbol{r}} = \frac{\boldsymbol{2}}{\boldsymbol{r}}$

Solution : $\nabla \cdot \left(\frac{1}{r^3}\vec{r}\right) = \nabla(\frac{1}{r^3}) \cdot \vec{r} + \frac{1}{r^3}(\nabla \cdot \vec{r})$ .... (1)

$$\nabla\left(\frac{1}{r^3}\right) = \vec{i}\frac{\partial\left(\frac{1}{r^3}\right)}{\partial x} + \vec{j}\frac{\partial\left(\frac{1}{r^3}\right)}{\partial y} + \vec{k}\frac{\partial\left(\frac{1}{r^3}\right)}{\partial z}$$

$$= \vec{i}\left(-\frac{3}{r^4}\right)\frac{\partial r}{\partial x} + \vec{j}\left(-\frac{3}{r^4}\right)\frac{\partial r}{\partial y} + \vec{k}\left(-\frac{3}{r^4}\right)\frac{\partial r}{\partial z}$$

$$= \vec{i}\left(-\frac{3}{r^4}\right)\frac{x}{r} + \vec{j}\left(-\frac{3}{r^4}\right)\frac{y}{r} + \vec{k}\left(-\frac{3}{r^4}\right)\frac{z}{r}$$

$$= \left(-\frac{3}{r^5}\right)(\vec{i}x + \vec{j}y + \vec{k}z) = \left(-\frac{3}{r^5}\right)\vec{r}$$

$$\therefore (1) \Rightarrow \nabla \cdot \left(\frac{1}{r^3}\vec{r}\right) = \left(-\frac{3}{r^5}\right)\vec{r} \cdot \vec{r} + \frac{1}{r^3}3 = \left(-\frac{3}{r^3}\right) + \frac{1}{r^3}3 = 0$$

$$\nabla \cdot \hat{r} = \nabla \cdot \frac{\vec{r}}{r} = \nabla \cdot \frac{1}{r}\vec{r}$$

$$\nabla \cdot \left(\frac{1}{r}\vec{r}\right) = \nabla\left(\frac{1}{r}\right) \cdot \vec{r} + \frac{1}{r}(\nabla \cdot \vec{r})$$

$$\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}\vec{r}\ (Verify)$$

$$\nabla \cdot \left(\frac{1}{r}\vec{r}\right) = \left(-\frac{1}{r^3}\vec{r}\right) \cdot \vec{r} + \frac{1}{r}3 = -\frac{1}{r^3} \times r^2 + \frac{3}{r} = \frac{2}{r}$$

Problem 22 : If $\vec{\boldsymbol{r}} = \boldsymbol{x\vec{i} + y\vec{j} + z\vec{k}}$ ***and*** $\boldsymbol{r = |\vec{r}|}$ ***show that*** $\boldsymbol{\nabla \cdot (f(r)\vec{r}) = r\,f'(r) + 3\,f(r)}$. ***Also if*** $\boldsymbol{\nabla \cdot (f(r)\vec{r}) = 0}$ ***show that*** $\boldsymbol{f(r) = \frac{c}{r^3}}$ ***where c is an arbitrary constant.***

Solution : $\nabla \cdot (f(r)\vec{r}) = \nabla(f(r)) \cdot \vec{r} + f(r)(\nabla \cdot \vec{r})$

$$\nabla(f(r)) = \frac{f'(r)}{r}\vec{r} \quad (Verify)$$

$$\nabla \cdot (f(r)\vec{r}) = \frac{f'(r)}{r}\vec{r}\cdot\vec{r} + f(r)3 = \mathrm{r}f'(\mathrm{r}) + \mathrm{f}(\mathrm{r})3$$

Also, given $\nabla \cdot (f(r)\vec{r}) = 0$

Thus, $rf'(r) + f(r)3 = 0$

$$rf'(r) = -f(r)3$$

$$\frac{f'(r)}{f(r)} = -\frac{3}{r}$$

Integrating both sides with respect to r

$$\int \frac{f'(r)}{f(r)}dr = -\int \frac{3}{r}dr$$

$$\log f(r) = -3logr + logc = -logr^3 + logc = logc - logr^3 = \log c/r^3$$

Thus, $\log f(r) = log\frac{c}{r^3}$

Thus, $f(r) = \frac{c}{r^3}$ where c is an arbitrary constant.

Problem 23 : If 'a' is a constant vector and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then show that $\nabla \cdot \{(\vec{a}\cdot\vec{r})\vec{r}\} = 4(\vec{a}\cdot\vec{r})$.

Solution : Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z$$

Now, $\nabla \cdot \{(\vec{a}\cdot\vec{r})\vec{r}\} = \nabla \cdot \{(a_1x + a_2y + a_3z)\vec{r}\}$

$$= \nabla(a_1x + a_2y + a_3z)\cdot\vec{r} + (a_1x + a_2y + a_3z)(\nabla\cdot\vec{r}) \quad \text{..............} (1)$$

$$\nabla(a_1x + a_2y + a_3z) = \vec{i}\frac{\partial(a_1x + a_2y + a_3z)}{\partial x} + \vec{j}\frac{\partial(a_1x + a_2y + a_3z)}{\partial y} +$$

$$\vec{k}\frac{\partial(a_1x + a_2y + a_3z)}{\partial z}$$

$$= a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = \vec{a}$$

Also, $\nabla \cdot \vec{r} = 3$

$$\therefore (1) \Rightarrow \nabla \cdot \{(\vec{a} \cdot \vec{r})\vec{r}\} = \vec{a} \cdot \vec{r} + 3(\vec{a} \cdot \vec{r}) = 4(\vec{a} \cdot \vec{r})$$

Problem 24: Find the value of 'a' if $\vec{A} = (axy - z^2)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 - axz)\vec{k}$ ***is irrotational.***

Solution : Given, $\vec{A}$ is irrotational.

$$\therefore \nabla \times \vec{A} = \vec{0}$$

$$\therefore \nabla \times \vec{A} = (2y - 2y)\vec{i} - (-az + 2z)\vec{j} + (2x - ax)\vec{k}$$

$$\nabla \times \vec{A} = \vec{0} \Rightarrow (2y - 2y)\vec{i} - (-az + 2z)\vec{j} + (2x - ax)\vec{k} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\therefore 2x - ax = 0$$

$$\therefore a = 2.$$

Problem 25 : Show that the following vector point functions are irrotational.

(i) $(4xy - z^3)\vec{i} + 2x^2\vec{j} - 3xz^2\vec{k}$

(ii) $(3x^2 + 2y^2 + 1)\vec{i} + (4xy - 3y^2z - 3)\vec{j} + (2 - y^3)\vec{k}$

(iii) $(y^2 + 2xz^2 - 1)\vec{i} + 2xy\vec{j} + 2x^2z\vec{k}$

Problem 26 : Show that the following vector $(y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}$ ***is both solenoidal and irrotational.***

Problem 27 : If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ ***for all f(r), show that*** $\nabla \times \{f(r)\vec{r}\} = \vec{0}$.

Solution : We have, $\nabla \times (f(r)\vec{r}) = \nabla(f(r)) \times \vec{r} + f(r)(\nabla \times \vec{r})$

$$\nabla(f(r)) = \frac{f'(r)}{r}\vec{r} \quad (Verify)$$

$$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0}$$

Thus, $\nabla \times (f(r)\vec{r}) = \frac{f'(r)}{r}\vec{r} \times \vec{r} + f(r)(\vec{0}) = \vec{0} + \vec{0} = \vec{0}.$

Problem 28 : : If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ show that $\nabla \times (r^n\vec{r}) = \vec{0}$.

Solution : $\nabla \times (r^n\vec{r}) = \nabla(r^n) \times \vec{r} + r^n(\nabla \times \vec{r})$ .... (1)

$$\nabla(r^n) = \vec{i}\frac{\partial(r^n)}{\partial x} + \vec{j}\frac{\partial(r^n)}{\partial y} + \vec{k}\frac{\partial(r^n)}{\partial z}$$

$$= \vec{i}nr^{n-1}\frac{\partial \mathrm{r}}{\partial x} + \vec{j}nr^{n-1}\frac{\partial \mathrm{r}}{\partial y} + \vec{k}nr^{n-1}\frac{\partial \mathrm{r}}{\partial z}$$

$$= \vec{i}nr^{n-1}\frac{x}{r} + \vec{j}nr^{n-1}\frac{y}{r} + \vec{k}nr^{n-1}\frac{z}{r}$$

$$= nr^{n-2}(\vec{i}x + \vec{j}y + \vec{k}z) = nr^{n-2}\vec{r}$$

$$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0}$$

$\therefore$ (1) => $\nabla \times (r^n\vec{r}) = \nabla(r^n) \times \vec{r} + r^n(\nabla \times \vec{r}) = nr^{n-2}\vec{r} \times \vec{r} + r^n\,(\vec{0}) = \vec{0}$

Problem 29 : Show that $\nabla \times \hat{r} = \vec{0}$.

(Hint : Put $\hat{r} = \frac{1}{r}\vec{r}$)

Problem 30 : If $\vec{v} = \vec{w} \times \vec{r}$ where w is a constant vector and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Show that $\frac{1}{2}curl\vec{v} = \vec{w}$.

Solution : Let $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$

$$curl\vec{v} = \sum \vec{i} \times \frac{\partial\vec{v}}{\partial x} = \sum \vec{i} \times \frac{\partial(\vec{w} \times \vec{r})}{\partial x} = \sum \vec{i} \times \left(\vec{w} \times \frac{\partial(\vec{r})}{\partial x}\right)$$

$$= \sum \vec{i} \times (\vec{w} \times \vec{i}) \quad \left[\text{ Since, } \frac{\partial\vec{r}}{\partial x} = \vec{i}\right]$$

$$= \sum[(\vec{i}\cdot\vec{i})\vec{w} - (\vec{i}\cdot\vec{w})\vec{i}] = \sum(\vec{i}\cdot\vec{i})\vec{w} - \sum(\vec{i}\cdot\vec{w})\vec{i}$$

$$= [(\vec{i}\cdot\vec{i})\vec{w} + (\vec{j}\cdot\vec{j})\vec{w} + (\vec{k}\cdot\vec{k})\vec{w}] - [(\vec{i}\cdot\vec{w})\vec{i} + (\vec{j}\cdot\vec{w})\vec{j} + (\vec{k}\cdot\vec{w})\vec{k}]$$

$$= (\vec{w} + \vec{w} + \vec{w}) - (w_1\vec{i} + w_2\vec{j} + w_3\vec{k}) = 3\vec{w} - \vec{w} = 2\vec{w}$$

Thus, $curl\vec{v} = 2\vec{w}$; Hence, $\frac{1}{2}curl\vec{v} = \vec{w}$.

Problem 31 : If 'a' is a constant vector, show that

(i) $\boldsymbol{\nabla} \times \{(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{r}})\vec{\boldsymbol{r}}\} = \vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}}$

(ii) $\boldsymbol{\nabla} \cdot \{\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}}\} = \mathbf{0}$

Solution :

(i) Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$$\nabla \times \{(\vec{a} \cdot \vec{r})\vec{r}\} = \{\nabla(\vec{a} \cdot \vec{r})\} \times \vec{r} + (\vec{a} \cdot \vec{r})(\nabla \times \vec{r})$$

$$\vec{a} \cdot \vec{r} = a_1x + a_2y + a_3z$$

$$\nabla(\vec{a} \cdot \vec{r}) = \vec{i}\frac{\partial(a_1x + a_2y + a_3z)}{\partial x} + \vec{j}\frac{\partial(a_1x + a_2y + a_3z)}{\partial y} +$$

$$\vec{k}\frac{\partial(a_1x + a_2y + a_3z)}{\partial z}$$

$$= a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = \vec{a}$$

$$\nabla \times \vec{r} = \vec{0}$$

Thus,

$$\nabla \times \{(\vec{a} \cdot \vec{r})\vec{r}\} = \vec{a} \times \vec{r} + (\vec{a} \cdot \vec{r})\vec{0} = \vec{a} \times \vec{r}$$

(ii) $\nabla \cdot (\vec{a} \times \vec{r}) = -\nabla \cdot (\vec{r} \times \vec{a}) = -(\nabla \times \vec{r}) \times \vec{a} = -\vec{0} \times \vec{a} = 0.$

Problem 32 : If $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are irrotational, show that $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is solenoidal.

Solution : Given, $\vec{A}$ *and* $\vec{B}$ are irrotational

Therefore,

$$\nabla \times \vec{A} = \vec{0} \ \& \ \nabla \times \vec{B} = \vec{0}$$

To prove, $\vec{A} \times \vec{B}$ is solenoidal

i.e., to prove $\nabla \cdot (\vec{A} \times \vec{B})=0$

Now, $\nabla \cdot (\vec{A} \times \vec{B}) = (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A} = 0 - 0 = 0.$

Hence, $\vec{A} \times \vec{B}$ is solenoidal.

UNIT III

LAPLACIAN OPERATOR & LINE INTEGRALS

3.1 Laplacian Differential operator

The operator ∇^2 defined by $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian differential operator.

Laplace Equation

If $\emptyset$ is such that $\boldsymbol{\nabla^2}\emptyset = \mathbf{0}$ $(i.e., \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2} = 0)$ the $\emptyset$ is said to satisfied Laplace equation.

Harmonic function

A single valued function f(x,y,z) is said to be a harmonic function if its second order partial derivative exists and are continuous and if the function satisfies the Laplace equation $\boldsymbol{\nabla^2 f = 0}$.

Theorem 3.1 : If $\emptyset$ is a scalar point function then,

(i) ***Divergence of the gradient of $\emptyset$ is $\boldsymbol{\nabla^2}\emptyset$ $\boldsymbol{i.e.}, \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\emptyset) = \boldsymbol{\nabla^2}\emptyset$***

(ii) ***Curl of the gradient of $\emptyset$ vanishes. (i.e.,) $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\emptyset) = \vec{\mathbf{0}}$.***

Proof :

(i) We have, $\nabla\emptyset = \vec{i}\frac{\partial\emptyset}{\partial x} + \vec{j}\frac{\partial\emptyset}{\partial y} + \vec{k}\frac{\partial\emptyset}{\partial z}$

$$\nabla\cdot\nabla\emptyset = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\vec{i}\frac{\partial\emptyset}{\partial x} + \vec{j}\frac{\partial\emptyset}{\partial y} + \vec{k}\frac{\partial\emptyset}{\partial z}\right)$$

$$= \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2} = \nabla^2\emptyset$$

(ii) $$\nabla\times(\nabla\emptyset) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\emptyset}{\partial x} & \frac{\partial\emptyset}{\partial y} & \frac{\partial\emptyset}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2\emptyset}{\partial y\partial z} - \frac{\partial^2\emptyset}{\partial z\partial y}\right) - \vec{j}\left(\frac{\partial^2\emptyset}{\partial x\partial z} - \frac{\partial^2\emptyset}{\partial z\partial x}\right) + \vec{k}\left(\frac{\partial^2\emptyset}{\partial x\partial y} - \frac{\partial^2\emptyset}{\partial y\partial x}\right)$$

$$=0\vec{i}+0\vec{j}+0\vec{k}=\vec{0}$$

Theorem 3.2 : If $\vec{A}=A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$ where A_1, A_2, A_3 have continuous second partials, then

(i) ***divergence of a curl of a vector vanishes. i.e., $\nabla\cdot(\nabla\times\vec{A})=0$***

(ii) $\nabla\times(\nabla\times\vec{A})=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$

Proof : (i) $\nabla\times\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix}$

$$=\vec{i}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)-\vec{j}\left(\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial z}\right)+\vec{k}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)$$

$$\nabla\cdot(\nabla\times\vec{A})=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot\left[\vec{i}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)-\vec{j}\left(\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial z}\right)+\vec{k}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\right]$$

$$=\frac{\partial^2 A_3}{\partial x\partial y}-\frac{\partial^2 A_2}{\partial x\partial z}-\frac{\partial^2 A_3}{\partial x\partial y}+\frac{\partial^2 A_1}{\partial y\partial z}+\frac{\partial^2 A_2}{\partial x\partial z}-\frac{\partial^2 A_1}{\partial y\partial z}=0$$

ii) $\nabla\times\vec{A}=\vec{i}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)-\vec{j}\left(\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial z}\right)+\vec{k}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)$

$$\nabla\times(\nabla\times\vec{A})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z} & -\left(\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial z}\right) & \frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\end{vmatrix}$$

$$=\vec{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial z}\right)\right]-\vec{j}\ldots\ldots$$

$$=\sum\vec{i}\left[\frac{\partial^2 A_2}{\partial x\partial y}-\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_3}{\partial x\partial z}-\frac{\partial^2 A_1}{\partial z^2}\right]$$

$$=\sum\vec{i}\left[\frac{\partial^2 A_2}{\partial x\partial y}+\frac{\partial^2 A_3}{\partial x\partial z}-\left(\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right]$$

$$=\sum\vec{i}\left[\frac{\partial^2 A_2}{\partial x\partial y}+\frac{\partial^2 A_3}{\partial x\partial z}+\frac{\partial^2 A_1}{\partial x^2}-\frac{\partial^2 A_1}{\partial x^2}-\left(\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right]$$

(Add and subtract $\frac{\partial^2 A_1}{\partial x^2}$)

$$=\sum\vec{i}\left[\frac{\partial^2 A_2}{\partial x\partial y}+\frac{\partial^2 A_3}{\partial x\partial z}+\frac{\partial^2 A_1}{\partial x^2}-\left(\frac{\partial^2 A_1}{\partial x^2}+\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right]$$

$$= \sum \vec{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} + \frac{\partial A_1}{\partial x}\right) - \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right]$$

$$= \sum \vec{i}\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) - \sum \vec{i}\,(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2})$$

$$= \sum \vec{i}\frac{\partial}{\partial x}(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}.$$

Problem 1 : Show that $\boldsymbol{\nabla^2(logr) = \frac{1}{r^2}}$.

Solution : $\nabla^2(logr) = \frac{\partial^2}{\partial x^2}(logr) + \frac{\partial^2}{\partial y^2}(logr) + \frac{\partial^2}{\partial z^2}(logr)$ .... (1)

Now, $\frac{\partial^2}{\partial x^2}(logr) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(logr)\right] = \frac{\partial}{\partial x}\left[\frac{1}{r}\frac{\partial r}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{1}{r}\frac{x}{r}\right] = \frac{\partial}{\partial x}\left[\frac{x}{r^2}\right]$

$$= -\frac{2x^2}{r^4} + \frac{1}{r^2}$$

Similarly, $\frac{\partial^2}{\partial y^2}(logr) = -\frac{2y^2}{r^4} + \frac{1}{r^2}$ & $\frac{\partial^2}{\partial z^2}(logr) = -\frac{2z^2}{r^4} + \frac{1}{r^2}$.

Then Sub. all the values in (1) we get,

$$\nabla^2(logr) = -\frac{2x^2}{r^4} + \frac{1}{r^2} - \frac{2y^2}{r^4} + \frac{1}{r^2} - \frac{2z^2}{r^4} + \frac{1}{r^2}$$

$$= -\frac{2(x^2 + y^2 + z^2)}{r^4} + \frac{3}{r^2} = -\frac{2r^2}{r^4} + \frac{1}{r^2} = \frac{3}{r^2}.$$

Problem 2 : Show that $\boldsymbol{\nabla^2(r^n) = n(n+1)r^{n-2}}$.

Solution : $\nabla^2(r^n) = \frac{\partial^2}{\partial x^2}(r^n) + \frac{\partial^2}{\partial y^2}(r^n) + \frac{\partial^2}{\partial z^2}(r^n)$ .... (1)

$$\frac{\partial^2}{\partial x^2}(r^n) = nx^2(n-2)r^{n-4} + nr^{n-2}$$

Similarly $\frac{\partial^2}{\partial y^2}(r^n) = ny^2(n-2)r^{n-4} + nr^{n-2}$ &

$$\frac{\partial^2}{\partial z^2}(r^n) = nz^2(n-2)r^{n-4} + nr^{n-2}$$

$\therefore$ (1) => $\nabla^2(r^n) = nx^2(n-2)r^{n-4} + nr^{n-2} + ny^2(n-2)r^{n-4} + nr^{n-2} +$
$nz^2(n-2)r^{n-4} + nr^{n-2}$

$=nr^{n-2}$.

Problem 3 : Show that $(\vec{V}\times\nabla)\times\vec{r}=-2\vec{V}$.

[Hint: Let $\vec{V}=V_1\vec{i}+V_2\vec{j}+V_3\vec{k}$ & $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$

First find $\vec{V}\times\nabla$ and then find $(\vec{V}\times\nabla)\times\vec{r}$]

Problem 4 : Show that $(\vec{V}\cdot\nabla)\vec{V}=\frac{1}{2}\nabla V^2-\vec{V}\times(\nabla\times\vec{V})$.

Problem 5 : If $\emptyset$ is a harmonic function then show that $\nabla\emptyset$ is solenoidal.

Solution : Given, $\emptyset$ is a harmonic function.

$$\therefore We\ have, \nabla^2\emptyset=0$$

To prove, $\nabla\emptyset$ is solenoidal

i.e., to prove, $\nabla\cdot\nabla\emptyset=0$

$$\nabla\cdot\nabla\emptyset=\nabla^2\emptyset=0$$

Hence, $\nabla\emptyset$ is solenoidal.

Problem 6: Show that $\nabla\cdot(\emptyset\nabla\psi-\psi\nabla\emptyset)=\emptyset\nabla^2\psi-\psi\nabla^2\emptyset$.

Solution : LHS$=\nabla\cdot(\emptyset\nabla\psi-\psi\nabla\emptyset)=\nabla\cdot(\emptyset\nabla\psi)-\nabla\cdot(\psi\nabla\emptyset)$ .... (1)

Consider, $\nabla\cdot(\emptyset\nabla\psi)$

Let $\vec{A}=\nabla\psi$

Now, $\nabla\cdot(\emptyset\vec{A})=\nabla\emptyset\cdot\vec{A}+\emptyset(\nabla\cdot\vec{A})=\nabla\emptyset\cdot\nabla\psi+\emptyset(\nabla\cdot\nabla\psi)=\nabla\emptyset\cdot\nabla\psi+\emptyset(\nabla^2\psi)$

Now, consider $\nabla\cdot(\psi\nabla\emptyset)$

Let $\vec{B}=\nabla\emptyset$

$$\nabla\cdot(\psi\vec{B})=\nabla\psi\cdot\vec{B}+\psi(\nabla\cdot\vec{B})=\nabla\psi\cdot\nabla\emptyset+\psi(\nabla\cdot\nabla\emptyset)=\nabla\psi\cdot\nabla\emptyset+\psi(\nabla^2\emptyset)$$

(1)=>, $\nabla\cdot(\emptyset\nabla\psi-\psi\nabla\emptyset)=\nabla\emptyset\cdot\nabla\psi+\emptyset(\nabla^2\psi)-\nabla\psi\cdot\nabla\emptyset-\psi(\nabla^2\emptyset)$

Thus, $\nabla\cdot(\emptyset\nabla\psi-\psi\nabla\emptyset)=\emptyset\nabla^2\psi-\psi\nabla^2\emptyset$.

Problem 7: Show that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$. ***Also, show that if*** $\nabla^2 f(r) = 0$, ***then*** $f(r)=\frac{\alpha}{r} + \beta$, ***where*** α ***and*** β ***are arbitrary constants.***

Solution : We have, $r^2 = x^2 + y^2 + z^2$ and $\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$.

$$\nabla^2 f(r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(r)$$

$$= \frac{\partial^2}{\partial x^2}(f(r)) + \frac{\partial^2}{\partial y^2}(f(r)) + \frac{\partial^2}{\partial z^2}(f(r))$$

$$\frac{\partial}{\partial x}(f(r)) = f'(r)\frac{\partial r}{\partial x} = \frac{f'(r)x}{r}$$

$$\frac{\partial^2}{\partial x^2}(f(r)) = \frac{\partial^2}{\partial x^2}\left(\frac{f'(r)x}{r}\right) = \frac{r\left[f'(r)1 + xf''(r)\frac{\partial r}{\partial x}\right] - f'(r)x\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r\left[f'(r)1 + xf''(r)\frac{x}{r}\right] - f'(r)x\frac{x}{r}}{r^2}$$

$$= \frac{[f'(r)r + f''(r)x^2\,] - \frac{x^2 f'(r)}{r}}{r^2}$$

$$= \frac{rx^2 f''(r) + r^2 f'(r) - x^2 f'(r)}{r^3}$$

$$\nabla^2 f(r) = \frac{rx^2 f''(r) + r^2 f'(r) - x^2 f'(r) + ry^2 f''(r) + r^2 f'(r) - y^2 f'(r) + rz^2 f''(r) + r^2 f'(r) - z^2 f'(r)}{r^3}$$

$$= \frac{\{rf''(r)[x^2 + y^2 + z^2] + 3r^2 f'(r) - f'(r)[x^2 + y^2 + z^2]\}}{r^3}$$

$$= \frac{\{rr^2 f''(r) + 3r^2 f'(r) - f'(r)r^2\}}{r^3}$$

$$= \frac{r^3 f''(r) + 2r^2 f'(r)}{r^3}$$

$$= f''(r) + \frac{2}{r} f'(r)$$

$$\nabla^2 f(r) = 0 \Rightarrow f''(r) + \frac{2}{r} f'(r) = 0$$

$$\Rightarrow \frac{f''(r)}{f'(r)} + \frac{2}{r} = 0$$

Integrating, $\int \frac{f''(r)}{f'(r)} dr + \int \frac{2}{r} dr = c$

$$\Rightarrow log f'(r) + 2\, log r = log k$$

$$\Rightarrow log f'(r) + \ log r^2 = log k$$

$$\Rightarrow log f'(r) r^2 = log k$$

$$\Rightarrow f'(r) r^2 = k$$

$$\Rightarrow f'(r) = \frac{k}{r^2}$$

Integrating, $\int f'(r) dr = \int \frac{k}{r^2} dr + \beta$

$$\Rightarrow f(r) = -\frac{k}{r} + \beta \Rightarrow f(r) = \frac{\alpha}{r} + \beta.$$

Problem 8: Show that $(\nabla\emptyset) \times (\nabla\psi)$ ***is solenoidal.***

Solution : To prove $(\nabla\emptyset) \times (\nabla\psi)$ is solenoidal.

i.e., to prove that $\nabla \cdot \big((\nabla\emptyset) \times (\nabla\psi)\big) = 0$

Let $\vec{A} = \nabla\emptyset \ \ and \ \vec{B} = \nabla\psi$.

$$\therefore \nabla \cdot \big((\nabla\emptyset) \times (\nabla\psi)\big) = \nabla \cdot \big(\vec{A} \times \vec{B}\big)$$

$$= \vec{B} \cdot \big(\nabla \times \vec{A}\big) - \vec{A} \cdot \big(\nabla \times \vec{B}\big)$$

$$= \vec{B} \cdot \vec{0} - \vec{A} \cdot \vec{0} = 0.$$

Hence, $(\nabla\emptyset) \times (\nabla\psi)$ is solenoidal.

Problem 9: Prove the following:

(i) $\boldsymbol{\nabla} \cdot \emptyset(\boldsymbol{\nabla\psi}) = (\boldsymbol{\nabla}\emptyset) \cdot (\boldsymbol{\nabla\psi}) + \emptyset(\boldsymbol{\nabla}^2\boldsymbol{\psi})$

(ii) $\boldsymbol{\nabla} \cdot \emptyset(\boldsymbol{\nabla}\emptyset) = (\boldsymbol{\nabla}\emptyset)^2 + \emptyset(\boldsymbol{\nabla}^2\emptyset)$

(iii) $\boldsymbol{\nabla} \times \emptyset(\boldsymbol{\nabla\psi}) = (\boldsymbol{\nabla}\emptyset) \times (\boldsymbol{\nabla\psi})$

(iv) $\boldsymbol{\nabla} \times \emptyset(\boldsymbol{\nabla}\emptyset) = \vec{\boldsymbol{0}}$.

Solution : (i) We know that $\nabla \cdot \emptyset\vec{A} = (\nabla\emptyset) \cdot \vec{A} + \emptyset(\nabla \cdot \vec{A})$

$$\therefore \nabla \cdot \emptyset(\nabla\psi) = (\nabla\emptyset) \cdot (\nabla\psi) + \emptyset(\nabla \cdot \nabla\psi)$$

$$= (\nabla\emptyset) \cdot (\nabla\psi) + \emptyset(\nabla^2\psi)$$

(ii) We know that $\nabla \cdot \emptyset\vec{A} = (\nabla\emptyset) \cdot \vec{A} + \emptyset(\nabla \cdot \vec{A})$

$$\therefore \nabla \cdot \emptyset(\nabla\emptyset) = (\nabla\emptyset) \cdot (\nabla\emptyset) + \emptyset(\nabla \cdot \nabla\emptyset)$$

$$= (\nabla\emptyset)^2 + \emptyset(\nabla^2\emptyset)$$

(iii) We know that $\nabla \times \emptyset\vec{A} = (\nabla\emptyset) \times \vec{A} + \emptyset(\nabla \times \vec{A})$

$$\therefore \nabla \times \emptyset(\nabla\psi) = (\nabla\emptyset) \times (\nabla\psi) + \emptyset(\nabla \times \nabla\psi)$$

$$= (\nabla\emptyset) \times (\nabla\psi) + \emptyset(\vec{0})$$

$$= (\nabla\emptyset) \times (\nabla\psi)$$

(iv) We know that $\nabla \times \emptyset\vec{A} = (\nabla\emptyset) \times \vec{A} + \emptyset(\nabla \times \vec{A})$

$$\therefore \nabla \times \emptyset(\nabla\emptyset) = (\nabla\emptyset) \times (\nabla\emptyset) + \emptyset(\nabla \times \nabla\emptyset)$$

$$= \vec{0} + \emptyset(\vec{0})$$

$$= \vec{0}.$$

3.2 LINE INTEGRALS

Suppose that C is an arc of a curve oriented from the end point A to the end point B. Let $\emptyset$ be a scalar point function defined at all points on C. Divide C arbitrarily into m parts. Let $P_0P_1, P_1P_2, P_2P_3, \dots\dots\dots\dots, P_{m-1}P_m$ be the m arcs. Also let Δs_i be the arcual distance of P_i from P_{i-1} and Q_i a point on the arc $P_{i-1}P_i$. It is noted the $\Delta s_i, i = 1, 2, 3, \dots\dots\dots\dots, m$, are all positive. Now the limit of the sum

$$\sum_{i=1}^{m} \emptyset(Q_i)\Delta s_i = \emptyset(Q_1)\Delta s_1 + \emptyset(Q_2)\Delta s_2 + \cdots \dots\dots\dots. + \emptyset(Q_m)\Delta s_m \quad \text{………. (3.1.1)}$$

As $m \to \infty$ and $max\Delta s_i \to 0$, if it exists, is called the line or curvilinear integral of $\emptyset$ along C and is denoted by

$$\int_C \emptyset\, ds \quad or \quad \int_{C(AB)} \emptyset\, ds,$$

C is called the path of integration. Now it is evident that

$$\int_{C(AB)} \emptyset\, ds = -\int_{C(BA)} \emptyset\, ds$$

When the equation of C is given in intrinsic form $r = x(s)\vec{\imath} + y(s)\vec{\jmath} + z(s)\vec{k}, 0 \le s \le l$, A and B correspond respectively to s=0 and $s = l$. Now, for points on C, $\emptyset$ is a function of s only and so (3.1.1) is

$$\int_{C(AB)} \emptyset\, ds = \int_0^l \emptyset\{x(s), y(s), z(s)\}\, ds.$$

Further, if the equation of C is given in terms of a parameter u in the form $= x(u)\vec{\imath} + y(u)\vec{\jmath} + z(u)\vec{k}$, $u_1 \le u \le u_2$ where u_1 *and* u_2 correspond to A and B, then

$$\int_{C(AB)} \emptyset\, ds = \int_{u_1}^{u_2} \emptyset\{x(u), y(u), z(u)\}\frac{ds}{du}\, du.$$

3.2.1 CONSERVATIVE FIELD AND SCALAR POTENTIAL

If a vector field f is such that there exists the scalar point function φ such that $\vec{f} = \nabla\varphi$, the $\vec{f}$ is said to be conservative field and φ is said to be scalar potential.

Theorem 1:

In a conservative field $f, \int_C f \cdot dr = 0$ where C is any simple closed curve.

Theorem 3.2.1 : The necessary and sufficient condition for the line integral $\int_{C(A_1,A_2)} \vec{f} \cdot dr$ ***to be independent of the path of integration is the existence of a scalar point function*** φ ***such that*** $\vec{f} = \nabla\varphi$.

Proof : Necessary part

Given that the line integral depends on the end points alone.

We have to prove the existence of a scalar function φ such that $\vec{f} = \nabla\varphi$.

Suppose that $\vec{f}$ is defined in D and that the symbol (A_1P) denotes any curve in D joining A_1 and P. If $P(x, y, z)$ is a variable point in D, then the line integral $\int_{(A_1P)} \vec{f} \cdot d\vec{r}$ ……….(1) depends on P and not on the curve (A_1P).

Hence, the integral (1) defines a scalar point function in D.

Let this function be denoted by $\varphi(P)$, that is, $\varphi(x, y, z)$.

Hence $\varphi(x,y,z) = \int_{(A_1P)} \vec{f} \cdot d\vec{r} = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \vec{f} \cdot d\vec{r}$

Let $P_1(x+\Delta x, y, z)$ be a point in D in the neighbourhood of P. Then

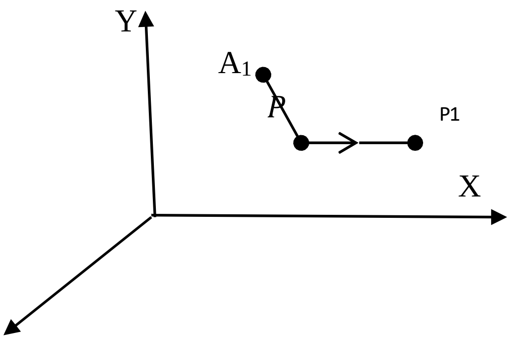
$$\varphi(x+\Delta x, y, z) = \int_{(A_1P)} \vec{f} \cdot d\vec{r} = \int_{(A_1PP_1)} \vec{f} \cdot d\vec{r}$$

$$= \int_{(A_1P)} \vec{f} \cdot d\vec{r} + \int_{(PP_1)} \vec{f} \cdot d\vec{r}$$

$$= \varphi(x,y,z) + \int_{(PP_1)} \vec{f} \cdot d\vec{r}$$

$$\Rightarrow \varphi(x+\Delta x, y, z) - \varphi(x,y,z) = \int_{(PP_1)} \vec{f} \cdot d\vec{r} \; \ldots\ldots\ldots\ldots (2)$$

Since the line integral is independent of the path of integration in D, we shall evaluate the integral in (2) by choosing (PP_1) as the straight line joining $P \,\&\, P_1$. This path is evidently parallel to the x-axis and along it $dy = 0$ and $dz = 0$.



Let $\vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$. Then (2) becomes

$$\varphi(x+\Delta x, y, z) - \varphi(x,y,z) = \int_{P}^{P_1} (f_1\vec{i} + f_2\vec{j} + f_3\vec{k}) \cdot dx\vec{i}$$

$$= \int_{P}^{P_1} f_1 dx$$

$$\therefore \frac{\varphi(x+\Delta x, y, z) - \varphi(x,y,z)}{\Delta x} = \frac{1}{\Delta x} \int_{(x,y,z)}^{(x+\Delta x,y,z)} f_1 dx$$

Taking limit as $P_1 \to P$, that is $\Delta x \to 0$, we get $\frac{\partial \varphi}{\partial x} = f_1$.

Similarly, $\frac{\partial \varphi}{\partial y} = f_2$ and $\frac{\partial \varphi}{\partial z} = f_3$.

Hence, $\vec{f} = f_1\vec{\imath} + f_2\vec{\jmath} + f_3\vec{k} = \vec{f} = \frac{\partial\varphi}{\partial x}\vec{\imath} + \frac{\partial\varphi}{\partial y}\vec{\jmath} + \frac{\partial\varphi}{\partial z}\vec{k} = \nabla\varphi$.

Sufficient Part:

Given that there exists a scalar function φ such that $\nabla\varphi = \vec{f}$. Let C be an arbitrary curve with end points (x_1, y_1, z_1) and (x_2, y_2, z_2).

Now $\int_C \vec{f} \cdot d\vec{r} = \int_C (\nabla\varphi) \cdot d\vec{r}$

$$= \int_C \left(\frac{\partial\varphi}{\partial x}\vec{\imath} + \frac{\partial\varphi}{\partial y}\vec{\jmath} + \frac{\partial\varphi}{\partial z}\vec{k}\right) \cdot \left(dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}\right)$$

$$= \int_C \left(\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz\right)$$

$$= \int_C d\varphi$$

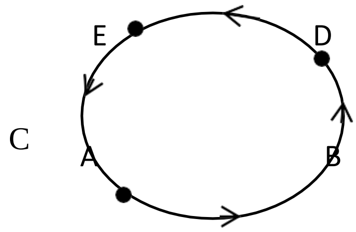
$$= [\varphi(x, y, z)]_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)}$$

$$= \varphi(x_2, y_2, z_2) - \varphi(x_1, y_1, z_1)$$

which is independent of the configuration of C. Hence the proof.

Theorem 3.2.2 : In a conservative field, $\vec{f}$, $\int_C \vec{f} \cdot d\vec{r} = 0$, where C is any simple closed curve.

Proof:



Let A, B, D, E be points on C taken in these in which C is oriented.

Now, $\int_C \vec{f} \cdot d\vec{r} = \int_{ABD} \vec{f} \cdot d\vec{r} + \int_{DEA} \vec{f} \cdot d\vec{r}$

$$= \int_{ABD} \vec{f} \cdot d\vec{r} + \int_{-AED} \vec{f} \cdot d\vec{r}$$

$$= \int_{ABD} \vec{f} \cdot d\vec{r} - \int_{AED} \vec{f} \cdot d\vec{r}$$

$$= \int_{ABD} \vec{f} \cdot d\vec{r} - \int_{ABD} \vec{f} \cdot d\vec{r}$$

$$= 0.$$

Theorem 3.2.3 : The necessary and sufficient condition for a vector field $\vec{f}$ to be conservative is that $\nabla \times \vec{f} = \vec{0}$.

Proof :

Necessity part : Given $\vec{f}$ to be conservative.

Let φ be the scalar potential.

Then, $\nabla \times \vec{f} = \nabla \times \nabla\varphi = \vec{0}$.

Sufficiency part :

Now, $\nabla \times \vec{f} = \vec{0}$.

To prove, $\vec{f}$ is conservative.

i.e., to prove there exists φ such that $\vec{f} = \nabla\varphi$.

Let $\vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$. Then $\nabla \times \vec{f} = \vec{0}$ implies

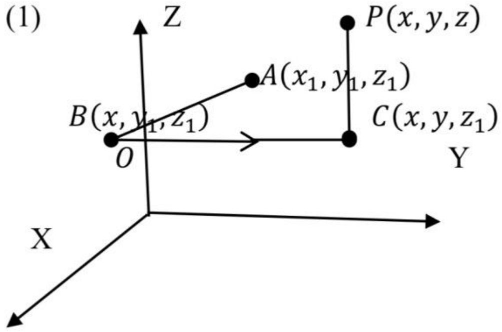
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \vec{0}$$

$$\Rightarrow \vec{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \vec{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \vec{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) = \vec{0}$$

$$\Rightarrow \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} = 0, \quad \frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} = 0 \text{ and } \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 0$$

$$\Rightarrow \frac{\partial f_3}{\partial y} = \frac{\partial f_2}{\partial z}, \quad \frac{\partial f_3}{\partial x} = \frac{\partial f_1}{\partial z} \text{ and } \frac{\partial f_2}{\partial x}\frac{\partial f_1}{\partial y}$$

.... (1)



$$\int_{ABCP} \vec{f} \cdot d\vec{r} = \int_{ABCP} [f_1(x, y, z)dx + f_2(x, y, z)dy + f_3(x, y, z)dz]$$

$$= \int_{AB} [f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz]$$

$$+ \int_{BC} [f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz]$$

$$+ \int_{CP} [f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz]$$

$$= \int_{x_1}^{x} f_1(x,y_1,z_1)dx + \int_{y_1}^{y} f_2(x,y,z_1)dy + \int_{z_1}^{z} f_3(x,y,z)dz$$

$$= \varphi(x,y,z), say \; [\, It \; is \; a \; scalar \; point \; function]$$

$$\therefore \frac{\partial\varphi}{\partial z} = 0 + 0 + \frac{\partial}{\partial z}\int_{z_1}^{z} f_3(x,y,z)dz = f_3(x,y,z)dz. \; [\frac{\partial}{\partial u}\int_{u_1}^{u} f(u)du = f(u)]$$

$$\frac{\partial\varphi}{\partial y} = 0 + f_2(x,y,z_1) + \int_{z_1}^{z} f_2(x,y,z)dz$$

$$= f_2(x,y,z_1) + \int_{z_1}^{z} \frac{\partial}{\partial z} f_2(x,y,z)dz \quad [By\ (1)]$$

$$= f_2(x,y,z_1) + [f_2(x,y,z)]_{z_1}^{z}$$

$$= f_2(x,y,z_1) + f_2(x,y,z) - f_2(x,y,z_1)$$

$$= f_2(x,y,z)$$

$$\frac{\partial\varphi}{\partial x} = f_1(x,y_1,z_1) + \int_{y_1}^{y} \frac{\partial}{\partial x} f_2(x,y,z_1)dy + \int_{z_1}^{z} \frac{\partial}{\partial x} f_3(x,y,z)dz$$

$$= f_1(x,y_1,z_1) + \int_{y_1}^{y} \frac{\partial}{\partial y} f_1(x,y,z_1)dy + \int_{z_1}^{z} \frac{\partial}{\partial z} f_1(x,y,z)dz \quad [By\ (1)]$$

$$= f_1(x,y_1,z_1) + f_1(x,y,z_1) - f_1(x,y_1,z_1) + f_1(x,y,z) - f_1(x,y,z_1)$$

$$= f_1(x,y,z).$$

Thus there exists a scalar function φ such that

$$\vec{f} = f_1\vec{\imath} + f_2\vec{\jmath} + f_3\vec{k} = \vec{f} = \frac{\partial\varphi}{\partial x}\vec{\imath} + \frac{\partial\varphi}{\partial y}\vec{\jmath} + \frac{\partial\varphi}{\partial z}\vec{k} = \nabla\varphi$$

$$\Longrightarrow \vec{f} = \nabla\varphi$$

Hence, $\vec{f}$ is a conservative field.

LINE INTEGRAL OF A CONSERVATIVE VECTOR

When $\vec{f}$ is conservative, the integral $\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} f \cdot dr$ can also be evaluated by using the scalar potential,

$$\vec{f} \cdot dr = \nabla\varphi \cdot dr$$

$$= \left(\vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}\right) \cdot (dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k})$$

$$= \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = d\varphi.$$

Problems

Find the value of the integral $\int_C \vec{A} \cdot dr$ where $\vec{A} = yz\vec{\imath} + zx\vec{\jmath} - xy\vec{k}$ in the following cases.

- ***(i)*** ***C is a curve whose parametric equations are $x = t, y = t^2, z = t^3$ drawn from $O(0,0,0)$ to $Q(2,4,8)$.***
- ***(ii)*** ***C is a curve obtain by joining O to $A(2,0,0)$, then A to $B(2,4,0)$ and then B to Q by straight lines.***
- ***(iii)*** ***C is the straight line joining O to Q.***

Solution:

(i) If $\vec{r}$ is the position vector, $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

$$\therefore \vec{r} = t\vec{\imath} + t^2\vec{\jmath} + t^3\vec{k}$$

$$\vec{A} = yz\vec{\imath} + zx\vec{\jmath} - xy\vec{k}$$

$$= t^5\vec{\imath} + t^4\vec{\jmath} - t^3\vec{k}$$

$$t \text{ varies from } 0 \text{ to } 2.$$

$$dr = d(t)\vec{\imath} + d(t^2)\vec{\jmath} + d(t^3)\vec{k} = \vec{\imath} + 2t\vec{\jmath} + 3t^2\vec{k}$$

$$\vec{A} \cdot dr = (t^5\vec{\imath} + t^4\vec{\jmath} - t^3\vec{k}) \cdot \vec{\imath} + 2t\vec{\jmath} + 3t^2\vec{k}$$

$$= t^5 + 2t^5 - 3t^5$$

$$= 0.$$

Thus, $\int_C \vec{A} \cdot dr = \int_0^2 0dt = 0.$

(ii) $\int_C \vec{A} \cdot dr = \int_{OA} \vec{A} \cdot dr + \int_{AB} \vec{A} \cdot dr + \int_{BQ} \vec{A} \cdot dr$ .... (1)

The parametric equation of the straight line using vector equation is

$$\vec{r} = (1-t)\vec{a} + t\vec{b}\ , 0 \le t \le 1.$$

Equation of OA is $\vec{r} = (1-t)\vec{0} + t(2\vec{\imath}), 0 \le t \le 1.$

Thus $\vec{r} = 2t\vec{\imath}, 0 \le t \le 1.$

$$d\vec{r} = 2dt\vec{\imath}.$$

Along OA, $x = 2t, y = 0, z = 0.$

$$\vec{A} = yz\vec{\imath} + zx\vec{\jmath} - xy\vec{k} = 0\vec{\imath} + 0\vec{\jmath} - 0\vec{k} = \vec{0}$$

$$\vec{A} \cdot dr = 0$$

$$\int_{OA} \vec{A} \cdot dr = 0.$$

Equation of AB is $\vec{r} = (1-t)2\vec{\imath} + t(2\vec{\imath} + 4\vec{\jmath}), 0 \le t \le 1.$

Thus $\vec{r} = 2\vec{\imath} + 4t\vec{\jmath}, 0 \le t \le 1.$

$$d\vec{r} = 4dt\vec{\imath}.$$

Along AB, $x = 2, y = 4t, z = 0.$

$$\vec{A} = yz\vec{\imath} + zx\vec{\jmath} - xy\vec{k} = 0\vec{\imath} + 0\vec{\jmath} - 2(4t)\vec{k} = -8t\vec{k}$$

$$\vec{A} \cdot dr = 0$$

$$\therefore \int_{AB} \vec{A} \cdot dr = 0.$$

Equation of BQ is $\vec{r} = (1-t)(2\vec{\imath} + 4\vec{\jmath}) + t\left(2\vec{\imath} + 4\vec{\jmath} + 8\vec{k}\right), 0 \le t \le 1.$

Thus $\vec{r} = (2 - 2t + 2t)\vec{\imath} + (4 - 4t + 4t)\vec{\jmath} + 8t\vec{k}, 0 \le t \le 1.$

$$= 2\vec{\imath} + 4\vec{\jmath} + 8t\vec{k}$$

$$d\vec{r} = 8dt\vec{\imath}.$$

Along BQ, $x = 2, y = 4, z = 8t$.

$$\vec{A} = yz\vec{i} + zx\vec{j} - xy\vec{k} = 4(8t)\vec{i} + (8t)2\vec{j} - 2(4)\vec{k}$$

$$= 32t\vec{i} + 16t\vec{j} - 8\vec{k}.$$

$$\vec{A} \cdot dr = -64dt$$

$$\therefore \int_{BQ} \vec{A} \cdot dr = \int_0^1 -64dt$$

$$= -64[t]_0^1 = -64.$$

$$\therefore (1) \Rightarrow \int_C \vec{A} \cdot dr = 0 + 0 - 64 = -64.$$

(iii) C is the straight line joining O(0, 0, 0) to Q(2, 4, 8).

Cartesian equation of OQ is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$.

$\Rightarrow \frac{x-0}{2-0} = \frac{y-0}{4-0} = \frac{z-0}{8-0} = t$

$$\Rightarrow x = 2t, y = 4t, z = 8t, 0 \le t \le 1.$$

$$\vec{A} = yz\vec{i} + zx\vec{j} - xy\vec{k} = 4t(8t)\vec{i} + (8t)2t\vec{j} - 2t(4t)\vec{k}$$

$$= 32t^2\vec{i} + 16t^2\vec{j} - 8t^2\vec{k}$$

$$dr = d(2t)\vec{i} + d(4t)\vec{j} + d(8t)\vec{k} = 2dt\vec{i} + 4dt\vec{j} + 8dt\vec{k}$$

$$\vec{A} \cdot dr = \left(32t^2\vec{i} + 16t^2\vec{j} - 8t^2\vec{k}\right) \cdot (2dt\vec{i} + 4dt\vec{j} + 8dt\vec{k})$$

$$= 64t^2dt + 64t^2dt - 64t^2dt = 64t^2dt.$$

$$\int_C \vec{A} \cdot dr = \int_0^1 64t^2dt$$

$$= 64\left[\frac{t^3}{3}\right]_0^1 = \frac{64}{3}.$$

Problem 2: In the vector field $\vec{F} = z(x\vec{i} + y\vec{j} + z\vec{k})$ evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the following curves:

- ***(i)*** ***$x = t, y = t^2, z = t^3$ from (0, 0, 0) to (1, 1, 1)***
- ***(ii)*** ***rectilinear curve obtained by joining O(0, 0, 0), A(1, 0, 0), B(1, 1, 0), C(1, 1,1) by the straight lines.***
- ***(iii)*** ***straight line joining (0, 0, 0) to (1, 1, 1).***

Solution : $\vec{F} = z(x\vec{i} + y\vec{j} + z\vec{k})$ and $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

(i) Curve C is given by $x = t, y = t^2, z = t^3$

$$\therefore dx = dt, dy = 2tdt, dz = 3t^2dt$$

$$\vec{F} = t^3(t\vec{i} + t^2\vec{j} + t^3\vec{k}) = t^4\vec{i} + t^5\vec{j} + t^6\vec{k}$$

$$\vec{F} \cdot d\vec{r} = (t^4\vec{i} + t^5\vec{j} + t^6\vec{k}) \cdot (dt\vec{i} + 2tdt\vec{j} + 3t^2dt\vec{k})$$

$$= t^4dt + 2t^6dt + 3t^8dt$$

$$= (t^4 + 2t^6 + 3t^8)dt.$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (t^4 + 2t^6 + 3t^8)dt$$

$$= \left[\frac{t^5}{5} + \frac{2t^7}{7} + \frac{3t^9}{9}\right]_0^1$$

$$= \frac{1}{5} + \frac{2}{7} + \frac{1}{3} = \frac{86}{105}.$$

(ii) rectilinear curve obtained by joining O(0, 0, 0), A(1, 0, 0), B(1, 1, 0), C(1, 1,1) by the straight lines.

$\int_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{A} \cdot dr + \int_{AB} \vec{A} \cdot dr + \int_{BC} \vec{A} \cdot dr$........ (1)

$$\int_{OA} \vec{A} \cdot dr = 0, \int_{AB} \vec{A} \cdot dr = 0, \int_{BC} \vec{A} \cdot dr = \frac{1}{3}.$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \frac{1}{3}.$$

(iii) $x = t, y = t, z = t, 0 \leq t \leq 1.$ $\int_C \vec{F} \cdot d\vec{r} = 1.$

Problem 3: Find $\int_C \vec{F} \cdot d\vec{r}$

(i) $\vec{F} = 3xy\vec{i} - y^2\vec{j}$ ***and C is the curve*** $x = t, y = 2t^2$ ***from (0,0) to (1, 2);***

(ii) $\vec{F} = (3x^2 + 6y)\vec{i} - 14yzy\vec{j} + 20xz^2\vec{k}$ ***and C is the curve*** $x = t, y = t^2, z = t^3$ ***from (0, 0, 0) to (1, 1, 1).***

[Ans. (i) -7/6 (ii) 5]

Problem 4 : Show that the work done in moving a particle in the field of force $\vec{F} = 3xy\vec{i} + (x + y)\vec{j} - z\vec{k}$ ***along the curve*** $x = t + 1, y = t - 1, z = t^2$ ***from (2, 0, 1) to (4, 2, 9) is -12.***

Solution : Given $\vec{F} = 3xy\vec{i} + (x + y)\vec{j} - z\vec{k}$

Curve C is, $x = t+1, y = t-1, z = t^2$

x=2 & x=4 $\therefore t = x-1 \Rightarrow 2-1 = 1 \,\&\, t = 4-1 = 3.$

$$\vec{F} = 3(t+1)(t-1)\vec{\imath} + (t+1+t-1)\vec{\jmath} - t^2\vec{k}$$

$$= 3(t^2-1)\vec{\imath} + 2t\vec{\jmath} - t^2\vec{k}.$$

$$d\vec{r} = dt\vec{\imath} + dt\vec{\jmath} + 2tdt\vec{k}.$$

$$\vec{F}\cdot d\vec{r} = \left(3(t^2-1)\vec{\imath} + 2t\vec{\jmath} - t^2\vec{k}\right)\cdot(dt\vec{\imath} + dt\vec{\jmath} + 2tdt\vec{k})$$

$$= (3t^2 - 3 + 2t - 2t^3)dt.$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_1^3 (3t^2 - 3 + 2t - 2t^3)dt$$

$$= \left[\frac{3t^3}{3} - 3t + \frac{2t^2}{2} - \frac{2t^4}{4}\right]_1^3$$

$$= \left[27 - 9 + 9 - \frac{81}{2}\right] - \left[1 - 3 + 1 - \frac{1}{2}\right] = -12.$$

Problem 5: Show that, if C is the circle $\boldsymbol{x = 3cost, y = 3sint, z = 0, then \oint_C[(2x - y + z)\vec{\imath} + (x + y - z^2)\vec{\jmath} + (3x - 2y + 4z)\vec{k}]\cdot d\vec{r} = 18\pi}$.

Solution : Given $\vec{F} = \left[(2x - y + z)\vec{\imath} + (x + y - z^2)\vec{\jmath} + (3x - 2y + 4z)\vec{k}\right]$

Curve C is $x = 3cost, y = 3sint, z = 0$

$$\therefore dx = -3sintdt, \qquad dy = 3costdt, \qquad dz = 0$$

$$\vec{F} = (6cost - 3sint + 0)\vec{\imath} + (3cost + 3sint - 0)\vec{\jmath} + (9cost - 6sint + 0)\vec{k}$$

$$= (6cost - 3sint)\vec{\imath} + (3cost + 3sint)\vec{\jmath} + (9cost - 6sint)\vec{k}$$

$$d\vec{r} = -3sint\, dt\vec{\imath} + 3costdt\vec{\jmath} + 0dt\vec{k}.$$

$$\vec{F}\cdot d\vec{r} = ((6cost - 3sint)\vec{\imath} + (3cost + 3sint)\vec{\jmath} + (9cost - 6sint)\vec{k})$$

$$\cdot(-3sint\, dt\vec{\imath} + 3costdt\vec{\jmath})$$

$$= [-18sintcost + 9sin^2t]dt + [9cos^2t + 9sintcost]dt$$

$$= [-18 sint cost + 9 sin^2 t + 9 cos^2 t + 9 sint cost]dt$$

$$= [-9 sint cost + 9]dt$$

$$= 9\left[1 - \frac{sin2t}{2}\right]dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 9\left[1 - \frac{sin2t}{2}\right]dt$$

$$= 9\left[t + \frac{cos2t}{4}\right]_0^{2\pi}$$

$$= 9\left[2\pi + \frac{cos4\pi}{4}\right] - 9\left[0 + \frac{cos0}{4}\right]$$

$$= 9\left[2\pi + \frac{1}{4}\right] - 9\left[\frac{1}{4}\right] = 18\pi.$$

Problem 6: Show that, if C is the semi-circle $x^2 + y^2 = 1$, z=0, drawn from (1, 0, 0) to (-1, 0,0) through (0, 1, 0), then $\int_C (x^3 - y^3)dy = \frac{3\pi}{8}$.

Solution : Curve C is $x = cost, y = sint, z = 0, 0 \leq t \leq \pi$

$$dx = -sintdt, dy = cost\, dt$$

$$\int_C (x^3 - y^3)dy = \int_0^{\pi} (cos^3 t - sin^3 t)costdt$$

$$= \int_0^{\pi} (cos^4 t - sin^3 tcost)\, dt$$

$$= \int_0^{\pi} ((cos^2 t)^2 - sin^2 tsintcost)\, dt$$

$$= \int_0^{\pi} \left[\left(\frac{1 + cos2t}{2}\right)^2 - \left(\frac{1 - cos2t}{2}\right)\frac{sin2t}{2}\right] dt$$

$$= \int_0^{\pi} \left[\frac{1}{4}(1 + cos^2 2t + 2cos2t - sin2t + sin2tcos2t)\ \right] dt$$

$$= \int_0^{\pi} \left[\frac{1}{4}\left(1 + \frac{1 + cos4t}{2} + 2cos2t - sin2t + \frac{sin4t}{2}\right)\right] dt$$

$$= \int_0^{\pi} \left[\frac{1}{4}\left(1 + \frac{1}{2} + \frac{cos4t}{2} + 2cos2t - sin2t + \frac{sin4t}{2}\right)\right] dt$$

$$= \left[\frac{1}{4}\left(t + \frac{1}{2}t + \frac{sin4t}{8} + \frac{2sin2t}{2} + \frac{cos2t}{2} - \frac{cos4t}{8}\right)\right]_0^{\pi}$$

$$= \frac{3\pi}{8}.$$

Problem 7: Evaluate the following integrals if C is the arc of the parabola $y^2 = 2x, z = 0$ from (0, 0, 0) to (1/2,0, 0) and $\emptyset = 3y - z, \vec{A} = 2xy\vec{i} + y\vec{j} + xz\vec{k}$:

(i) $\int_C \emptyset ds$ ***(ii)*** $\int_C \vec{A} ds$ ***(iii)*** $\int_C \vec{A} \cdot dr$ ***(iv)*** $\int_C \vec{A} \times dr$

Solution: Given $\vec{A} = 2xy\vec{i} + y\vec{j} + xz\vec{k}$

Curve C is $y^2 = 2x$

Put y=t, $t^2 = 2x \Rightarrow x = \frac{t^2}{2}, z = 0, 0 \le t \le 1.$

$$\vec{A} = t^3\vec{i} + t\vec{j}$$

$$\vec{r} = \frac{t^2}{2}\vec{i} + t\vec{j}$$

$$\frac{d\vec{r}}{dt} = t\vec{i} + \vec{j}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{t^2 + 1}$$

$$\therefore ds = \left|\frac{d\vec{r}}{dt}\right| dt = \sqrt{t^2 + 1}dt$$

$$\int_C \emptyset ds = \int_0^1 (3t - 1)\sqrt{t^2 + 1}dt$$

$$= \int_1^2 \frac{3}{2}\sqrt{h}dt \quad [put\ t^2 + 1 = h, 2tdt = dh, t = 0 \Rightarrow h = 1, t = 1 \Rightarrow h = 2]$$

$= 2\sqrt{2} - 1.$

(ii) $\int_C \vec{A} ds = \int_0^1 (t^3\vec{\imath} + t\vec{\jmath})\sqrt{t^2+1} dt = \vec{\imath}\left[\frac{1}{15} 2(\sqrt{2}+1)\right] + \vec{\jmath}\left[\frac{1}{3}(2\sqrt{2}-1)\right].$

(iii) $\int_C \vec{A} \cdot dr = \int_0^1 (t^4 + t) dt = 7/10$

(iv) $\int_C \vec{A} \times dr = \int_0^1 \vec{k}(t^3 - t^2) dt = \vec{k}\left(-\frac{1}{12}\right).$

Problem 8: ***If*** $\vec{A} = y\vec{\imath} - z\vec{\jmath} + x\vec{k}$ ***and C is the arc of the curve*** $\vec{r} = cosu\vec{\imath} + sinu\vec{\jmath} + u\vec{k}$ ***where the parameter u takes the values from 0 to*** 2π***, show that the following:***

(i) $\int_C \vec{A} ds = -2\sqrt{2}\vec{\jmath}$ ***(ii)*** $\int_C \vec{A} \cdot dr = -\pi$ ***(iii)*** $\int_C \vec{A} \times d\vec{r} = -(2\pi^2 + \pi)\vec{\imath} + 2\pi\vec{\jmath}.$

Problem 9: ***Show that if C is the ellipse*** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$ ***then*** $\int_C \left(\frac{b^2x^2}{a^2} + \frac{a^2y^2}{b^2}\right)^{1/2} ds = (a^2 + b^2)\pi.$

[Hint : Parametric equations of ellipse $x = acost, y = bsint, 0 \le t \le 2\pi, ds = \left|\frac{d\vec{r}}{dt}\right| dt = \sqrt{a^2sin^2t + b^2cos^2t} dt$].

Problem 10: ***Show that if C is the triangle with vertices (0, 0, 0), (1, 0, 0), (1, 1,0), then*** $\int_C (y^2dx + x^2dy) = \frac{1}{3}.$

Problem 11: ***Show that the integral of*** $\vec{F} = (3x^2 + 6xy)\vec{\imath} + (3x^2 - y^3)\vec{\jmath}$ ***is independent of path of integration. Find*** $\int_C \vec{F} \cdot d\vec{r}$ ***along any curve joining (0, 0) and (1, 2).***

Solution : For proving $\vec{F}$ is conservative field, we show that $\nabla \times \vec{F} = \vec{0}$.

Now, $\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 + 6xy & 3x^2 - y^3 & 0 \end{vmatrix} = \vec{0}.$

Hence, $\vec{F}$ is conservative field.

Hence, $\vec{F} = \nabla\emptyset$ for some scalar $\emptyset$. By theorem, $\int_C \vec{F} \cdot d\vec{r}$ is independent of path C.

Cartesian equation of line joining (0, 0) and (1, 2) is $\frac{x-0}{1-0} = \frac{y-0}{2-0} = t$

$$\Rightarrow x = t, y = 2t, 0 \le t \le 1.$$

$$\vec{F} = (15t^2)\vec{\imath} + (3t^2 - 8t^3)\vec{\jmath}$$

$$d\vec{r} = dt\vec{i} + 2dt\vec{j}.$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left((15t^2)\vec{i} + (3t^2 - 8t^3)\vec{j}\right)\cdot (dt\vec{i} + 2dt\vec{j})$$

$$= \int_0^1 (21t^2 - 16t^3)dt$$

$$= \left[21\frac{t^3}{3} - 16\frac{t^4}{4}\right]_0^1$$

$$= 7 - 4 = 3.$$

UNIT-IV

SURFACE INTEGRAL & VOLUME INTEGRAL

4.1 Surface Integrals

Given a scalar point function ϕ defined at all points of a surface S, the limit of the sum

$$\phi(Q_1)\Delta S_1 + \phi(Q_2)\Delta S_2 + \cdots + \phi(Q_m)\Delta S_m \qquad (1)$$

as $m \to \infty$ and $max\, \Delta Si \to 0$, where $\Delta S_1, \Delta S_2, \dots, \Delta S_m$ are the areas of the m small arbitrary bits of surfaces into which S is subdivided and Q_i is a point in ΔS_i, is called the surface integral of ϕ on S and is denoted by $\iint_s \phi\, dS$. This integral is scalar.

Similarly, if we consider $\vec{f}$, a vector point function defined on S, instead of ϕ, we get the surface integral of $\vec{f}$ on S to be $\iint_s \vec{f}\, dS = \vec{\imath} \iint_s f_1\, dS + \vec{\jmath} \iint_s f_2\, dS + \vec{k} \iint_s f_3\, dS$, where $\vec{f} = f_1\, \vec{\imath} + f_2\, \vec{\jmath} + f_3\, \vec{k}$. If ϕ and $\vec{A}$ are twopoint functions defined on S, then some more point functions can be defined on S such as $\phi\, \vec{n},\ \vec{A}.\vec{n},\ \vec{A}\times\vec{n}$. Correspondingly, we have the surface integrals $\iint_s \phi\, \vec{n}\, dS,\ \iint_s \vec{A}.\vec{n}\, dS,\ \iint_s \vec{A}\times\vec{n}\, dS$.

We have $d\vec{S} = \vec{n} dS$, the above integrals can be written as

$$\iint_S \phi\, d\vec{S}, \iint \vec{A}.\, d\vec{S}, \iint_S \vec{A} \times d\vec{S}.$$

In physical application, the integral $\iint_S \vec{A}\,.\, d\vec{S}$ is called the **flux** of $\vec{A}$ through S.

Evaluation of surface integrals

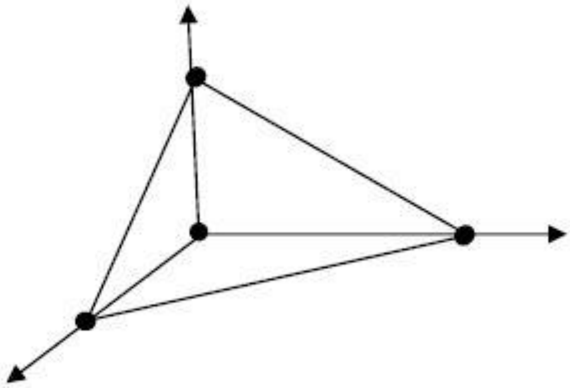
$$\iint_S \phi\, dS = \iint_{R_{xy}} \phi \frac{dxdy}{|\vec{n}.\vec{k}|}$$

$$\iint_S \phi\, dS = \iint_{R_{yz}} \phi \frac{dydz}{|\vec{n}.\vec{\imath}|}$$

$$\iint_S \phi\, dS = \iint_{R_{zx}} \phi \frac{dzdx}{|\vec{n}.\vec{\jmath}|}$$

where R_{xy}, R_{yz}, R_{zx} are the projection of the surface S on the XOY, YOZ, XOZ planes.

Problem 1. ***Evaluate the integral*** $\iint_S (\vec{A}.\vec{n})\, dS$ ***if*** $\vec{A} = 4y\, \vec{\imath} + 18z\, \vec{\jmath} - x\, \vec{k}$ ***and*** S ***is the surface of the portion of the plane*** $3x + 2y + 6z = 6$ ***contained in the first octant.***



Solution. *Given* $\vec{A} = 4y\,\vec{\imath} + 18z\,\vec{\jmath} - x\,\vec{k}$ *and* $\phi = 3x + 2y + 6z - 6$.

Now, $\iint_S (\vec{A}.\vec{n})\, dS = \iint_{R_{xy}} (\vec{A}.\vec{n}) \frac{dxdy}{|\vec{n}.\vec{k}|}$ where $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{\imath} + \frac{\partial\phi}{\partial y}\vec{\jmath} + \frac{\partial\phi}{\partial z}\vec{k} = 3\vec{\imath} + 2\vec{\jmath} + 6\vec{k}$$

$$|\nabla\phi| = \sqrt{9 + 4 + 36} = 7$$

$$\therefore\ \vec{n} = \frac{1}{7}\left(3\vec{\imath} + 2\vec{\jmath} + 6\vec{k}\right)$$

Hence, $\vec{n}.\vec{k} = \frac{6}{7}$ and

$$\vec{A}.\vec{n} = \frac{1}{7}(12y + 36z - 6x) = \frac{6}{7}(2y + 6z - x) = \frac{6}{7}(2y + 6 - 3x - 2y - x)$$

$$= \frac{6}{7}(6 - 4x).$$

On the xoy-plane, $z = 0$.

$\therefore\ 3x + 2y = 6 \Rightarrow y = \frac{6-3x}{2}$.

$\therefore\ y$ varies from 0 to $3 - \frac{3x}{2}$ and x varies from 0 to 2.

$$\therefore \iint_S (\vec{A}.\vec{n})\, dS = \int_{x=0}^{2}\int_{y=0}^{3-\frac{3x}{2}} \frac{6}{7}(6 - 4x)\frac{dxdy}{6/7}$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{3-\frac{3x}{2}} (3 - 2x)\, dxdy$$

$$= 2\int_{x=0}^{2} (3 - 2x) \int_{y=0}^{3-\frac{3x}{2}} dydx$$

$$= 2\int_{x=0}^{2} (3 - 2x)\, [y]_0^{3-\frac{3x}{2}} dx$$

$$= 2\int_{x=0}^{2} (3 - 2x)\left(3 - \frac{3x}{2}\right) dx$$

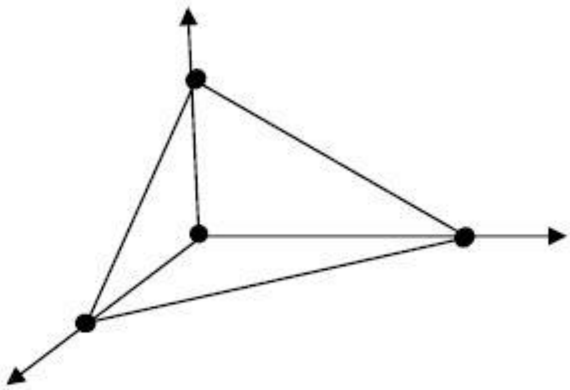
$$= 2\int_{x=0}^{2}\left(9 - \frac{9x}{2} - 6x + 3x^2\right)dx$$

$$= 2\left[9x - \frac{9x^2}{4} - 3x^2 + x^3\right]_0^2$$

$$= 2[18 - 9 - 12 + 8 - 0]$$

$$= 2(26 - 21) = 10$$

Problem 2. ***Evaluate the integral*** $\iint_S (\vec{A}.\vec{n})\, dS$ ***if*** $\vec{A} = (x + y^2)\vec{\imath} - 2x\,\vec{\jmath} + 2yz\,\vec{k}$***and*** S***is the surface of the portion of the plane*** $2x + y + 2z = 6$***contained in the first octant.***



Solution. Given $\vec{A} = (x + y^2)\vec{\imath} - 2x\,\vec{\jmath} + 2yz\,\vec{k}$ and $\phi = 2x + y + 2z - 6$.

Now, $\iint_S (\vec{A}.\vec{n})\, dS = \iint_{R_{xy}} (\vec{A}.\vec{n}) \frac{dxdy}{|\vec{n}.\vec{k}|}$ where $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{\imath} + \frac{\partial\phi}{\partial y}\vec{\jmath} + \frac{\partial\phi}{\partial z}\vec{k} = 2\vec{\imath} + \vec{\jmath} + 2\vec{k}$$

$$|\nabla\phi| = \sqrt{4 + 1 + 4} = 3$$

$$\therefore\ \vec{n} = \frac{1}{7}(2\vec{\imath} + \vec{\jmath} + 2\vec{k})$$

Hence, $\vec{n}.\vec{k} = \frac{2}{3}$ and

$$\vec{A}.\vec{n} = \frac{1}{3}(2x + 2y^2 - 2x + 4yz)$$

$$= \frac{2}{3}(y^2 + 2yz)$$

$$= \frac{2}{3}\big(y^2 + y(6 - 2x - y)\big)$$

$$= \frac{2}{3}(y^2 + 6y - 2xy - y^2)$$

$$= \frac{2}{3}(6x - 2xy)$$

$$= \frac{4}{3}(3 - x)y$$

On the xoy-plane, $z = 0$.

$\therefore\ 2x + y = 6 \ \Rightarrow\ y = 6 - 2x.$

$\therefore\ y$ varies from 0 to $6 - 2x$ and x varies from 0 to 3.

$$\therefore \iint_S (\vec{A}.\vec{n})\, dS = \int_{x=0}^{3}\int_{y=0}^{3-2x} \frac{4}{3}(3 - x)y\frac{dxdy}{2/3}$$

$$= 2\int_{x=0}^{3}\int_{y=0}^{3-2x}(3 - x)y\, dxdy$$

$$= 2\int_{x=0}^{3}(3 - x)\int_{y=0}^{3-2x} ydydx$$

$$= 2\int_{x=0}^{3}(3 - x)\left[\frac{y^2}{2}\right]_0^{6-2x} dx$$

$$= \int_{x=0}^{3}(3 - x)\,(36 - 24 + 4x^2)dx$$

$$= 4\int_{x=0}^{3}(3 - x)\,(9 - 6x + x^2)dx$$

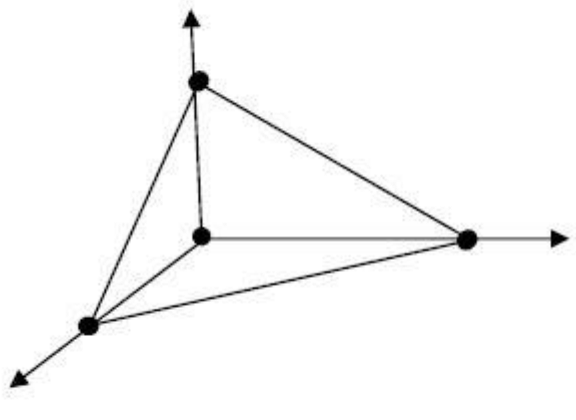
$$= 4\int_{x=0}^{3}(27 - 18x + 3x^2 - 9x + 6x^2 - x^3)\, dx$$

$$= 4\left[27x - 27\frac{x^2}{2} + 3x^3 - \frac{1}{4}x^4\right]_0^3$$

$$= 4\left[81 - 27\left(\frac{9}{2}\right) + 81 - \frac{81}{4}\right]$$

$$= 4(81) - 6(81) + 4(81) - 81 = 81$$

Problem 3. ***Evaluate the integral*** $\iint_S (\vec{A}.\vec{n})\, dS$ ***if*** $\vec{A} = 18z\,\vec{i} - 12\,\vec{j} + 3y\,\vec{k}$ ***and S is the surface of the portion of the plane*** $2x + 3y + 6z = 12$ ***contained in the first octant.***



Solution. Given$\vec{A} = 18z\,\vec{\imath} - 12\,\vec{\jmath} + 3y\,\vec{k}$ and $\phi = 2x + 3y + 6z - 12$.

Now, $\iint_S (\vec{A}.\vec{n})\,dS = \iint_{R_{xy}} (\vec{A}.\vec{n}) \frac{dxdy}{|\vec{n}.\vec{k}|}$ where $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{\imath} + \frac{\partial\phi}{\partial y}\vec{\jmath} + \frac{\partial\phi}{\partial z}\vec{k} = 2\vec{\imath} + 3\vec{\jmath} + 6\vec{k}$$

$$|\nabla\phi| = \sqrt{4+9+36} = 7$$

$$\therefore\ \vec{n} = \frac{1}{7}(2\vec{\imath} + 3\vec{\jmath} + 6\vec{k})$$

Hence, $\vec{n}.\vec{k} = \frac{6}{7}$ and

$$\therefore \iint_S (\vec{A}.\vec{n})\,dS = \int_{x=0}^{6}\int_{y=0}^{4-\frac{2x}{3}} \frac{12}{7}(3-x)\frac{dxdy}{6/7}$$

$$= 2\int_{x=0}^{6}\int_{y=0}^{4-\frac{2x}{3}} (3-x)\,dydx$$

$$= 2\int_{x=0}^{6} (3-x)\,[y]_0^{4-\frac{2x}{3}}\,dx$$

$$= 2\int_{x=0}^{6} (3-x)\left(4-\frac{2x}{3}\right)dx$$

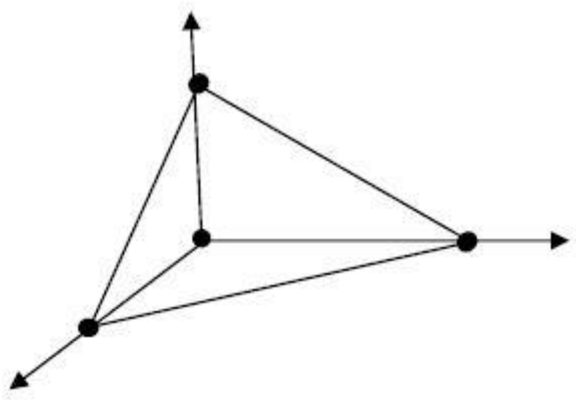
$$= 2\int_{x=0}^{6}\left(12-2x-4x+\frac{2x^2}{3}\right)dx$$

$$= 2\left[12 - 3x^2 + \frac{2}{9}x^3\right]_0^6$$

$$= 2[72 - 108 + 48]$$

$$= 2(12) = 24$$

Problem 4. ***Find the area of the surface*** S***of the portion of the plane*** $3x + 2y + 6z = 6$***contained in the first octant.***



Solution. Given $\phi = 3x + y + 2z - 6$.

Now, $\iint_S dS = \iint_{R_{xy}} \frac{dxdy}{|\vec{n}.\vec{k}|}$ where $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = 3\vec{i} + 2\vec{j} + 6\vec{k}$$

$$|\nabla\phi| = \sqrt{9+4+36} = 7$$

$$\therefore\ \vec{n} = \frac{1}{7}(3\vec{i} + 2\vec{j} + 6\vec{k})$$

Hence, $\vec{n}.\vec{k} = \frac{6}{7}$

On the xoy-plane, $z = 0$.

$\therefore\ 3x + 2y = 6 \Rightarrow y = 3 - \frac{3}{2}x.$

$\therefore\ y$ varies from 0 to $3 - \frac{3}{2}x$ and x varies from 0 to 2.

$$\therefore \text{Area} = \iint_S (\vec{A}.\vec{n})\,dS = \int_{x=0}^{2}\int_{y=0}^{3-\frac{3x}{2}} \frac{dxdy}{6/7}$$

$$= \frac{7}{6}\int_{x=0}^{2}\int_{y=0}^{3-\frac{3x}{2}} dydx$$

$$= \frac{7}{6}\int_{x=0}^{2} [y]_0^{3-\frac{3x}{2}}\,dx$$

$$= \frac{7}{6}\int_{x=0}^{2} \left(3 - \frac{3}{2}x\right)dx$$

$$= \frac{7}{6}\left[3x - \frac{3}{4}x^2\right]_0^2$$

$$= \frac{7}{6}[6-3] = \frac{7}{2}$$

Problem 5. ***Sis the surface of the portion of the plane $2x + 2y + z = 4$, in the first octant and $\vec{A} = 2x\,\vec{j} - xz\,\vec{k}$. Show that the following:***

i) $\iint_S (\vec{A}.\vec{n})\,dS = 4$

ii) $\iint_S (\vec{r}.\vec{n})\,dS = 8$

iii) $\iint_S \vec{A}\,dS = 4(2\,\vec{j} - \vec{k})$

iv) $\iint_S (\vec{A}\times\vec{n})\,dS = \frac{16}{3}\vec{i} - \frac{8}{3}\vec{j} + \frac{16}{3}\vec{k}$

Solution. ***i)*** Given $\vec{A} = 2x\,\vec{j} - xz\,\vec{k}$ and $\phi = 2x + 2y + z - 4$.

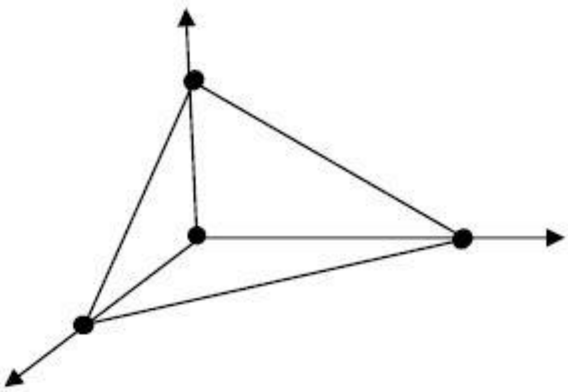
Now, $\iint_S (\vec{A}.\vec{n})\,dS = \iint_{R_{xy}} (\vec{A}.\vec{n})\frac{dxdy}{|\vec{n}.\vec{k}|}$ where $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = 2\vec{i} + 2\vec{j} + \vec{k}$$

$$|\nabla\phi| = \sqrt{4+4+1} = 3$$

$$\therefore\ \vec{n} = \frac{1}{7}(2\vec{i} + 2\vec{j} + \vec{k})$$

Hence, $\vec{n}.\vec{k} = \frac{1}{3}$ and



$$\vec{A}.\vec{n} = \frac{2}{3}(2x) + \frac{1}{3}(-xz)$$

$$= \frac{1}{3}(4x - xz)$$

$$= \frac{1}{3}(4 - z)x$$

$$= \frac{1}{3}(4 - 4 + 2x + 2y)x$$

$$= \frac{2}{3}(x^2 + xy)$$

On the xoy-plane, $z = 0$.

$\therefore\ 2x + 2y = 4 \Rightarrow y = 2 - x.$

$\therefore\ y$ varies from 0 to $2 - x$ and x varies from 0 to 2.

$$\therefore \iint_S (\vec{A}.\vec{n})\, dS = \int_{x=0}^{2}\int_{y=0}^{2-x}(x^2 + xy)\frac{dxdy}{1/3}$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x}(x^2 + xy)\, dydx$$

$$= 2\int_{x=0}^{2}\left[x^2y + x\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= 2\int_{x=0}^{2}\left[x^2(2-x) + \frac{x}{2}(4 - 4x + x^2)\right] dx$$

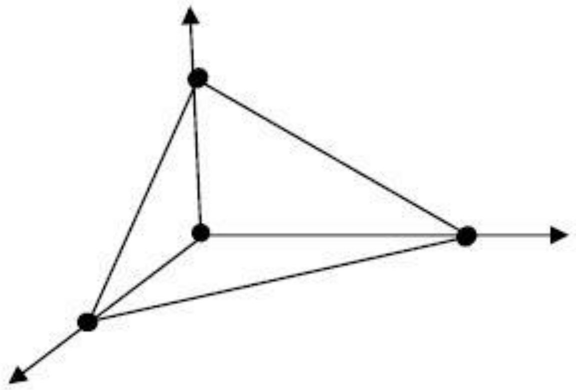
$$= 2\int_{x=0}^{2}\left(2x^2 - x^3 + 2x - 2x^2 + \frac{x^3}{2}\right) dx$$

$$= 2\int_{x=0}^{2}\left(2x - \frac{x^3}{2}\right)dx$$

$$= 2\left[x^2 - \frac{x^4}{8}\right]_0^2$$

$$= 2[4-2] = 4$$

ii) We have $\vec{r} = 2\vec{\imath} + 2\vec{\jmath} + \vec{k}$ and $\phi = 2x + 2y + z - 4$.



Now, $\iint_S (\vec{r}.\vec{n})\, dS = \iint_{R_{xy}} (\vec{r}.\vec{n}) \frac{dxdy}{|\vec{n}.\vec{k}|}$ where $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{\imath} + \frac{\partial\phi}{\partial y}\vec{\jmath} + \frac{\partial\phi}{\partial z}\vec{k} = 2\vec{\imath} + 2\vec{\jmath} + \vec{k}$$

$$|\nabla\phi| = \sqrt{4+4+1} = 3$$

$$\therefore\ \vec{n} = \frac{1}{7}\left(2\vec{\imath} + 2\vec{\jmath} + \vec{k}\right)$$

Hence, $\vec{n}.\vec{k} = \frac{1}{3}$ and

$$\vec{r}.\vec{n} = \frac{2}{3}(x) + \frac{2}{3}(y) + \frac{1}{3}(z)$$

$$= \frac{1}{3}(2x + 2y + z)$$

$$= \frac{1}{3}(2x + 2y + 4 - 2x - 2y) = \frac{4}{3}$$

On the xoy-plane, $z = 0$.

$\therefore\ 2x + 2y = 4 \ \Rightarrow\ y = 2 - x$.

$\therefore\ y$ varies from 0 to $2 - x$ and x varies from 0 to 2.

$$\therefore\ \iint_S (\vec{r}.\vec{n})\, dS = \int_{x=0}^{2}\int_{y=0}^{2-x} \frac{4}{3}\frac{dxdy}{1/3}$$

$$= 4\int_{x=0}^{2}\int_{y=0}^{2-x} dydx$$

$$= 4\int_{x=0}^{2} [y]_0^{2-x}\, dx$$

$$= 4\int_{x=0}^{2} [2-x]dx$$

$$= 4\left[2x - \frac{x^2}{2}\right]_0^2$$

$$= 4[4-2] = 8$$

iii) Given $\vec{A} = 2x\,\vec{j} - xz\,\vec{k}$ and $\phi = 2x + 2y + z - 4 = 0$.

$$\therefore\ \vec{A} = 2x\,\vec{j} - xz\,\vec{k}$$

$$= 2x\,\vec{j} - x(4 - 2x - 2y)\vec{k}$$

$$= 2x\,\vec{j} - (2x^2 + 2xy - 4x)\vec{k}$$

Now, $\iint_S \vec{A}\, dS = \iint_{R_{xy}} \vec{A}\frac{dxdy}{|\vec{n}.\vec{k}|}$ where $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = 2\vec{i} + 2\vec{j} + \vec{k}$$

$$|\nabla\phi| = \sqrt{4+4+1} = 3$$

$$\therefore\ \vec{n} = \frac{1}{3}\left(2\vec{i} + 2\vec{j} + \vec{k}\right)$$

Hence, $\vec{n}.\vec{k} = \frac{1}{3}$.

On the xoy-plane, $z = 0$.

$\therefore\ 2x + 2y = 4 \ \Rightarrow\ y = 2 - x$.

$\therefore\ y$ varies from 0 to $2 - x$ and x varies from 0 to 2.

$$\therefore \iint_S \vec{A}\, dS = \int_{x=0}^{2}\int_{y=0}^{2-x} \left[2x\vec{j} + (2x^2 + 2xy - 4x)\vec{k}\right]\frac{dxdy}{1/3}$$

$$= 3\int_{x=0}^{2}\int_{y=0}^{2-x} \left[2x\vec{j} + (2x^2 + 2xy - 4x)\vec{k}\right]dydx$$

$$= 3\int_{x=0}^{2} \left[2xj\vec{} + \left(2x^2y + 2x\frac{y^2}{2} - 4xy\right)\vec{k}\right]_0^{2-x} dx$$

$$= 3\int_{x=0}^{2} \left[2x(2-x)\vec{j} + \{2x^2(2-x) + x(4 - 4x + x^2) - 4x(2-x)\}\,\vec{k}\right]dx$$

$$= 3\int_{x=0}^{2} \left[(4x - 2x^2)\vec{j} + \{4x^2 - 2x^3 + 4x - 4x^2 + x^3 - 8x + 4x^2\}\vec{k}\right]dx$$

$$= 3\int_{x=0}^{2} \left[(4x - 2x^2)\vec{j} + (4x^2 - x^3 - 4x)\vec{k}\right]dx$$

$$= 3\left[\left(2x^2 - 2\frac{x^3}{3}\right)\vec{j} + \left(4\frac{x^3}{3} - \frac{x^4}{4} - 2x^2\right)\vec{k}\right]_0^2$$

$$= 3\left[\left(8 - \frac{16}{3}\right)\vec{j} + \left(\frac{32}{3} - 4 - 8\right)\vec{k}\right]$$

$$= (24 - 16)\vec{j} + (32 - 36)\vec{k}$$

$$= 8\vec{j} + 4\vec{k}$$

$$= 4(2\vec{j} - \vec{k})$$

iv) Given $\vec{A} = 2x\,\vec{j} - xz\,\vec{k}$ and $\phi = 2x + 2y + z - 4 = 0$.

$$\therefore\ \vec{A} = 2x\,\vec{j} - xz\,\vec{k}$$

$$= 2x\,\vec{j} - x(4 - 2x - 2y)\vec{k}$$

$$= 2x\,\vec{j} - (2x^2 + 2xy - 4x)\vec{k}$$

Now, $\iint_S (\vec{A} \times \vec{n})\, dS = \iint_{R_{xy}} (\vec{A} \times \vec{n}) \frac{dxdy}{|\vec{n}.\vec{k}|}$ where $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = 2\vec{i} + 2\vec{j} + \vec{k}$$

$$|\nabla\phi| = \sqrt{4 + 4 + 1} = 3$$

$$\therefore\ \vec{n} = \frac{1}{3}(2\vec{i} + 2\vec{j} + \vec{k})$$

Hence, $\vec{n}.\vec{k} = \frac{1}{3}$.

$$\text{Also } \vec{A} \times \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2x & 2x^2 + 2xy - 4x \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

$$= \vec{i}\left[\frac{2}{3}x - \frac{2}{3}(2x^2 + 2xy - 4x)\right] - \vec{j}\left[0 - \frac{2}{3}(2x^2 + 2xy - 4x)\right] + \vec{k}\left[0 - \frac{4}{3}x\right]$$

$$= \vec{i}\left[\frac{10}{3}x - \frac{4}{3}x^2 - \frac{4}{3}xy\right] - \vec{j}\left[\frac{4}{3}x^2 + \frac{4}{3}xy - \frac{8}{3}x\right] + \vec{k}\left[-\frac{4}{3}x\right]$$

On the xoy-plane, $z = 0$.

$\therefore\ 2x + 2y = 4\ \Rightarrow\ y = 2 - x$.

$\therefore\ y$ varies from 0 to $2 - x$ and x varies from 0 to 2.

$$\therefore \iint_S (\vec{A} \times \vec{n})\, dS$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\left[\vec{i}\left[\frac{10}{3}x - \frac{4}{3}x^2 - \frac{4}{3}xy\right] - \vec{j}\left[\frac{4}{3}x^2 + \frac{4}{3}xy - \frac{8}{3}x\right] + \vec{k}\left[-\frac{4}{3}x\right]\right]\frac{dxdy}{1/3}$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} \left[\vec{\imath}[10x - 4x^2 - 4xy] - \vec{\jmath}[4x^2 + 4xy - 8x] + \vec{k}[-4x\,]\right] dydx$$

$$= \int_{x=0}^{2} \left[\vec{\imath}[10xy - 4x^2y - 4xy^2] - \vec{\jmath}[4x^2y + 2xy^2 - 8xy] + \vec{k}[-4xy\,]\right]_0^{2-x} dx$$

$$= \int_{x=0}^{2} \{\vec{\imath}[10x(2 - x) - 4x^2(2 - x) - 2x(4 - 4x + x^2)]$$
$$- \vec{\jmath}[4x^2(2 - x) + 2^x(4 - 4x + x^2) - 8x(2 - x)]$$
$$+ \vec{k}[-4x(2 - x)]\}\, dx$$

$$= \int_{x=0}^{2} \{\vec{\imath}[20x - 10x^2 - 8x^2 + 4x^3 - 8x + 8x^2 - 2x^3]$$
$$- \vec{\jmath}[8x^2 - 4x^3 + 8x - 8x^2 + 2x^3 - 16x + 8x^2]$$
$$+ \vec{k}[-8x + 4x^2]\}\, dx$$

$$= \int_{x=0}^{2} \left[\vec{\imath}[12x - 10x^2 + 2x^3] - \vec{\jmath}[8x^2 - 2x^3 - 8x] + \vec{k}[-8x + 4x^2]\right] dx$$

$$= \left[\vec{\imath}\left[6x^2 - \frac{10}{3}x^3 + \frac{1}{2}x^4\right] - \vec{\jmath}\left[\frac{8}{3}x^3 - \frac{1}{2}x^4 - 4x^2\right] + \vec{k}\left[-4x^2 + \frac{4}{3}x^3\right]\right]_0^2$$

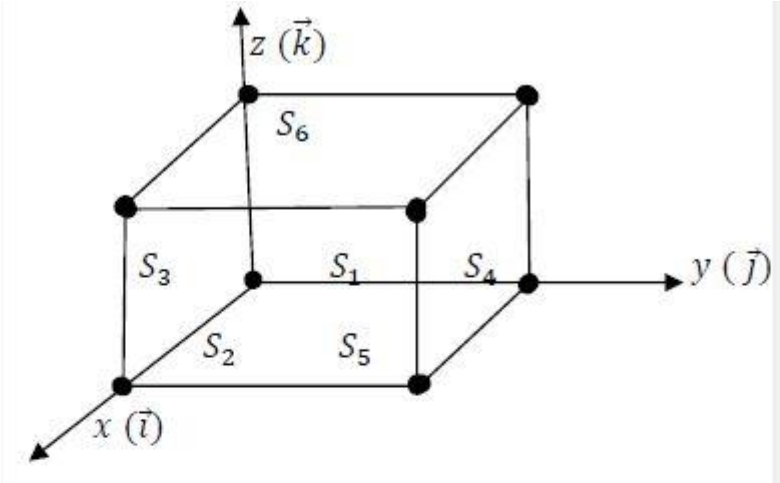
$$= \vec{\imath}\left[24 - \frac{80}{3} + 8\right] - \vec{\jmath}\left[\frac{64}{3} - 8 - 16\right] + \vec{k}\left[-16 + \frac{32}{3}\right]$$

$$= \vec{\imath}\left[\frac{96 - 80}{3}\right] - \vec{\jmath}\left[\frac{64 - 72}{3}\right] + \vec{k}\left[\frac{-48 + 32}{3}\right]$$

$$= \vec{\imath}\left[\frac{16}{3}\right] - \vec{\jmath}\left[\frac{8}{3}\right] + \vec{k}\left[\frac{16}{3}\right]$$

Problem 6. ***Show that*** $\iint_S (\vec{r}.\vec{n})\, dS = 3$, ***where Sis the surface of the cube bounded by the planes*** $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.$

Solution: Suppose the faces whose equations are $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ are respectively named $S_1, S_2, S_3, S_4, S_5, S_6$ and $\vec{n}$ denotes the unit normal to them drawn outwards.



On $S_1, x = 0, \vec{r} = y\vec{j} + z\vec{k}, \vec{n} = -\vec{i}$.

$\therefore$ y varies from 0 to 1 and z varies from 0 to 1.

$$\therefore \iint_{S_1} (\vec{r}.\vec{n})\, dS = \int_{y=0}^{1}\int_{z=0}^{1} (0)\, dydz = 0.$$

On $S_2, x = 1, \vec{r} = \vec{i} + y\vec{j} + z\vec{k}, \vec{n} = \vec{i}$.

$\therefore$ y varies from 0 to 1 and z varies from 0 to 1.

$$\therefore \iint_{S_2} (\vec{r}.\vec{n})\, dS = \int_{y=0}^{1}\int_{z=0}^{1} \frac{dydz}{|\vec{n}.\vec{i}|}$$

$$= \int_{y=0}^{1}\int_{z=0}^{1} dydz$$

$$= \int_{y=0}^{1} [z]_0^1\, dy$$

$$= \int_{y=0}^{1} dy = 1.$$

On $S_3, y = 0, \vec{r} = x\vec{i} + z\vec{k}, \vec{n} = -\vec{j}$.

$\therefore$ x varies from 0 to 1 and z varies from 0 to 1.

$$\therefore \iint_{S_3} (\vec{r}.\vec{n})\, dS = \int_{x=0}^{1}\int_{z=0}^{1} (0)\, dydz = 0.$$

On $S_4, y = 1, \vec{r} = x\vec{i} + \vec{j} + z\vec{k}, \vec{n} = \vec{j}$.

$\therefore$ x varies from 0 to 1 and z varies from 0 to 1.

$$\therefore \iint_{S_4} (\vec{r}.\vec{n})\, dS = \int_{x=0}^{1}\int_{z=0}^{1} (1)\frac{dxdz}{|\vec{n}.\vec{j}|}$$

$$= \int_{x=0}^{1}\int_{z=0}^{1} dxdz = 1.$$

On $S_5, z = 0, \vec{r} = x\vec{i} + y\vec{j}, \vec{n} = -\vec{k}$.

$\therefore$ x varies from 0 to 1 and y varies from 0 to 1.

$$\therefore \iint_{S_5} (\vec{r}.\vec{n})\, dS = \int_{x=0}^{1} \int_{y=0}^{1} (0)\, dxdy = 0.$$

On $S_6, z = 1, \vec{r} = x\vec{\imath} + y\vec{\jmath} + \vec{k}, \vec{n} = \vec{k}.$

$\therefore$ x varies from 0 to 1 and y varies from 0 to 1.

$$\therefore \iint_{S_6} (\vec{r}.\vec{n})\, dS = \int_{x=0}^{1} \int_{y=0}^{1} (1) \frac{dxdy}{|\vec{n}.\vec{k}|} = 1.$$

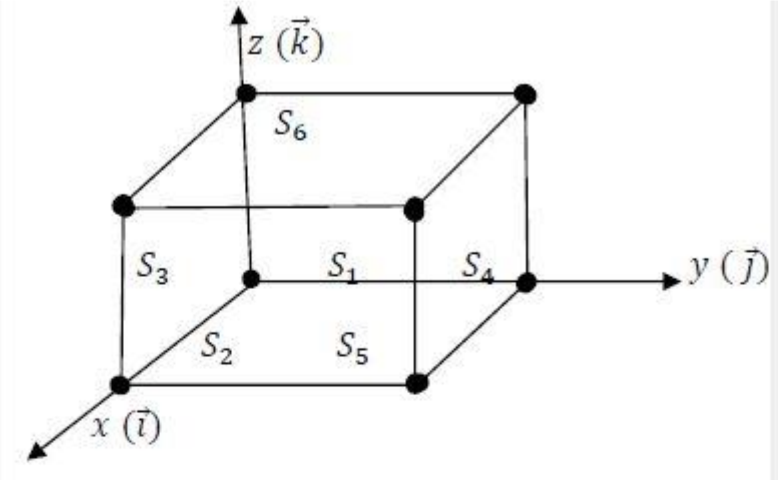
Hence,

$$\iint_{S} (\vec{r}.\vec{n})\, dS = \iint_{S_1} (\vec{r}.\vec{n})\, dS + \iint_{S_2} (\vec{r}.\vec{n})\, dS + \iint_{S_3} (\vec{r}.\vec{n})\, dS + \iint_{S_4} (\vec{r}.\vec{n})\, dS + \iint_{S_5} (\vec{r}.\vec{n})\, dS + \iint_{S_6} (\vec{r}.\vec{n})\, dS$$

$$= 0 + 1 + 0 + 1 + 0 + 1 = 3$$

Problem 7. ***Show that*** $\iint_S (\vec{F}.\vec{n})\, dS$ **if** $\vec{F} = (x + y)\vec{\imath} + x\vec{\jmath} + z\vec{k}$***and*** S ***is the surface of the cube bounded by the planes*** $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.$

Solution: Suppose the faces whose equations are $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ are respectively named $S_1, S_2, S_3, S_4, S_5, S_6$ and $\vec{n}$ denotes the unit normal to them drawn outwards.



On $S_1, x = 0, \vec{F} = y\vec{\jmath} + z\vec{k}, \vec{n} = -\vec{\imath}.$

$\therefore$ y varies from 0 to 1 and z varies from 0 to 1.

$$\therefore \iint_{S_1} (\vec{F}.\vec{n})\, dS = \int_{y=0}^{1} \int_{z=0}^{1} (-y) \frac{dydz}{|\vec{n}.\vec{\imath}|}$$

$$= -\int_{y=0}^{1} y\, dy$$

$$= \left[-\frac{y^2}{2}\right]_0^1$$

$$= -\frac{1}{2}.$$

On $S_2, x = 1, \vec{F} = (1+y)\vec{i} + \vec{j} + z\vec{k}, \vec{n} = \vec{i}.$

$\therefore$ y varies from 0 to 1 and z varies from 0 to 1.

$$\therefore \iint_{S_2} (\vec{F}.\vec{n})\, dS = \int_{y=0}^{1}\int_{z=0}^{1} (1+y)\frac{dydz}{|\vec{n}.\vec{i}|}$$

$$= \int_{y=0}^{1}\int_{z=0}^{1} (1+y)dydz$$

$$= \int_{y=0}^{1} (1+y)[z]_0^1\, dy$$

$$= \int_{y=0}^{1} (1+y)dy = 1$$

$$= \left[y + \frac{y^2}{2}\right]_0^1$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$

On $S_3, y = 0, \vec{F} = x\vec{i} + x\vec{j} + z\vec{k}, \vec{n} = -\vec{j}.$

$\therefore$ x varies from 0 to 1 and z varies from 0 to 1.

$$\therefore \iint_{S_3} (\vec{F}.\vec{n})\, dS = \int_{x=0}^{1}\int_{z=0}^{1} (-x)\, dxdz$$

$$= -\int_{x=0}^{1} x\, dx$$

$$= \left[-\frac{x^2}{2}\right]_0^1$$

$$= -\frac{1}{2}$$

On $S_4, y = 1, \vec{F} = (x+1)\vec{i} + x\vec{j} + z\vec{k}, \vec{n} = \vec{j}.$

$\therefore$ x varies from 0 to 1 and z varies from 0 to 1.

$$\therefore \iint_{S_4} (\vec{F}.\vec{n})\, dS = \int_{x=0}^{1}\int_{z=0}^{1} (x)\frac{dxdz}{|\vec{n}.\vec{j}|}$$

$$= \int_{x=0}^{1}\int_{z=0}^{1} x\, dxdz$$

$$= \int_{x=0}^{1} x[z]_0^1\, dx$$

$$= \int_{x=0}^{1} x dx$$

$$= \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$

On $S_5, z = 0, \vec{F} = (x + y)\vec{\imath} + x\vec{\jmath}, \vec{n} = -\vec{k}$.

$\therefore$ x varies from 0 to 1 and y varies from 0 to 1.

$$\therefore \iint_{S_5} (\vec{F}.\vec{n})\, dS = \int_{x=0}^{1}\int_{y=0}^{1} (0)\, dxdy = 0.$$

On $S_6, z = 1, \vec{r} = (x + y)\vec{\imath} + x\vec{\jmath} + \vec{k}, \vec{n} = \vec{k}$.

$\therefore$ x varies from 0 to 1 and y varies from 0 to 1.

$$\therefore \iint_{S_6} (\vec{F}.\vec{n})\, dS = \int_{x=0}^{1}\int_{y=0}^{1} (1)\frac{dxdy}{|\vec{n}.\vec{k}|}$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} dydx = 1$$

Hence,

$$\iint_S (\vec{F}.\vec{n})\, dS = \iint_{S_1} (\vec{F}.\vec{n})\, dS + \iint_{S_2} (\vec{F}.\vec{n})\, dS + \iint_{S_3} (\vec{F}.\vec{n})\, dS + \iint_{S_4} (\vec{F}.\vec{n})\, dS$$

$$+ \iint_{S_5} (\vec{F}.\vec{n})\, dS + \iint_{S_6} (\vec{F}.\vec{n})\, dS$$

$$= -\frac{1}{2} + \frac{3}{2} - \frac{1}{2} + \frac{1}{2} + 0 + 1 = 2$$

Cylindrical Surface:

Problem 8. ***Evaluate*** $\iint_S (\vec{A}.\vec{n})\, dS$ ***if*** $\vec{A} = yz\vec{\imath} + 2y^2\vec{\jmath} + xz^2\vec{k}$ ***and*** S ***is the surface of the cylinder*** $x^2 + y^2 = 9$ ***contained in the first octant between the planes*** $z = 0$ ***and*** $z = 2$.

Solution. Let the projection R of S on the xoz-plane be the rectangle $OABC$.

R is given by $0 \le x \le 3$ and $0 \le z \le 2$.

Given $\vec{A} = yz\vec{\imath} + 2y^2\vec{\jmath} + xz^2\vec{k}$ and ϕ is $x^2 + y^2 = 9$

Now, $\iint_S (\vec{A}.\vec{n}) dS = \iint_{R_{xz}} (\vec{A}.\vec{n}) \frac{dxdz}{|\vec{n}.\vec{\jmath}|}$ where $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

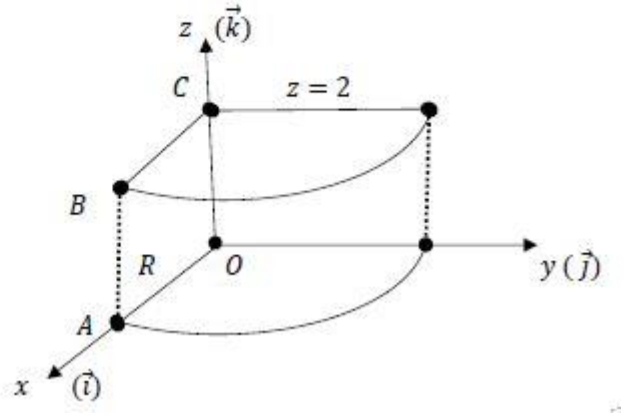
$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{\imath} + \frac{\partial\phi}{\partial y}\vec{\jmath} + \frac{\partial\phi}{\partial z}\vec{k} = 2x\vec{\imath} + 2y\vec{\jmath}$$

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2} = \sqrt{4(9)} = 6$$

$$\therefore\ \vec{n} = \frac{1}{6}(2x\vec{i} + 2y\vec{j})$$

Hence, $\vec{n}.\vec{j} = \frac{y}{3}$ and

$$\vec{A}.\vec{n} = \frac{1}{3}(xyz) + \frac{1}{3}(2y^3) = \frac{1}{3}y(xz + 2y^2)$$



$$\iint_S (\vec{A}.\vec{n})\, dS = \int_{x=0}^{3}\int_{z=0}^{2} \frac{1}{3}y(xz + 2y^2)\frac{dxdz}{y/3}$$

$$= \int_{x=0}^{3}\int_{z=0}^{2} (xz + 2y^2)dzdx$$

$$= \int_{x=0}^{3}\int_{z=0}^{2} (xz + 18 - 2x^2)dzdx$$

$$= \int_{x=0}^{3}\left[x\frac{z^2}{2} + 18z - 2x^2z\right]_0^2 dx$$

$$= \int_{x=0}^{3}[2x + 36 - 4x^2]dx$$

$$= \left[x^2 + 36x - 4\frac{x^3}{3}\right]_0^3$$

$$= 9 + 108 - 36 = 81$$

Problem 9. ***Evaluate*** $\iint_S(\vec{A}.\vec{n})\,dS$ ***if*** $\vec{A} = z\vec{i} + x\vec{j} + y^2\vec{k}$ ***and S is the surface of the cylinder*** $x^2 + y^2 = 1$ ***contained in the first octant between the planes*** $z = 0$ ***and*** $z = 2$.

Solution. Let the projection R of S on the xoz-plane be the rectangle $OABC$.

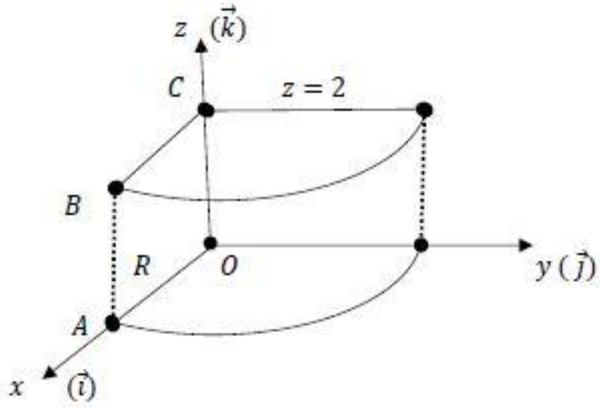
R is given by $0 \le x \le 1$ and $0 \le z \le 2$.

Given $\vec{A} = z\vec{i} + x\vec{j} + y^2\vec{k}$ and ϕ is $x^2 + y^2 = 1$.

Now, $\iint_S(\vec{A}.\vec{n})dS = \iint_{R_{xy}}(\vec{A}.\vec{n})\frac{dxdz}{|\vec{n}.\vec{j}|}$ where $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = 2x\vec{i} + 2y\vec{j}$$

$$|\nabla\phi| = \sqrt{4x^2+4y^2} = \sqrt{4(1)} = 2$$



$$\therefore\ \vec{n} = \frac{1}{2}(2x\vec{i}+2y\vec{j}) = x\vec{i}+y\vec{j}$$

Hence, $\vec{n}.\vec{j} = y$ and

$\vec{A}.\vec{n} = xz + xy$

$$\iint_S (\vec{A}.\vec{n})\,dS = \int_{x=0}^{1}\int_{z=0}^{2}(xz+xy)\frac{dxdz}{|y|}$$

$$= \int_{x=0}^{1}\int_{z=0}^{2}\left(xz + x\sqrt{1-x^2}\right)\frac{dzdx}{\sqrt{1-x^2}}$$

$$= \int_{x=0}^{1}\int_{z=0}^{2}\left(\frac{xz}{\sqrt{1-x^2}} + x\right)dzdx$$

$$= \int_{x=0}^{1}\left[\frac{x}{\sqrt{1-x^2}}\frac{z^2}{2} + xz\right]_0^2 dx$$

$$= \int_{x=0}^{1}\left[\frac{x}{\sqrt{1-x^2}}2 + 2x\right]dx \quad [1-x^2 = t \Rightarrow 2xdx = -dt]$$

$$= \int_{x=0}^{1}\frac{2x}{\sqrt{1-x^2}}dx + \int_{x=0}^{1}2xdx$$

$$= \int_{t=1}^{0}\frac{-dt}{\sqrt{t}} + [x^2]_0^1$$

$$= \int_{t=1}^{0} t^{-1/2}\,dt + 1$$

$$= \left[2t^{1/2}\right]_0^1 + 1$$

$$= 2 + 1 = 3.$$

Problem 10. Evaluate $\iint_S (\vec{A}.\vec{n})\,dS$ ***if*** $\vec{A} = 4x\vec{i} - 2y\vec{j} + z^2\vec{k}$***and*** S ***is the surface of the region bounded by*** $x^2+y^2=4, z=0, z=3.$

Solution. Let S be divided into three parts S_1, S_2, S_3 where S_1 denotes the curved surface, S_2 denotes the plane surface in the $z = 0$ plane and S_3 denotes the plane surface in the $z = 3$ plane. Cylindrical coordinates of the cylinder are (r, θ, z).

Here, $r = 2$. Hence $x = r\cos\theta = 2\cos\theta$, $y = r\sin\theta = 2\sin\theta$, $z = z$.

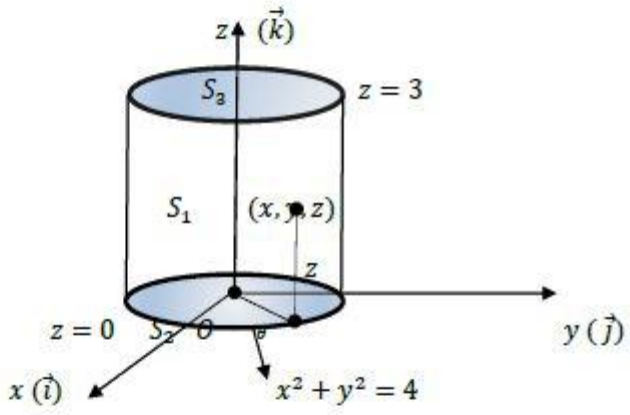
$dS = r\,d\theta\,dz = 2\,d\theta\,dz$. Now $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = 2x\vec{i} + 2y\vec{j}$$

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2} = \sqrt{4(4)} = 4$$

$$\therefore\ \vec{n} = \frac{1}{4}(2x\vec{i} + 2y\vec{j}) = \frac{1}{2}(x\vec{i} + y\vec{j})$$

$$\vec{A}.\vec{n} = \frac{1}{2}(4x^2 - 2y^3).$$



$$\iint_{S_1} (\vec{A}.\vec{n})\,dS = \int_{\theta=0}^{2\pi}\int_{z=0}^{3} \frac{1}{2}(4x^2 - 2y^3)\,2d\theta\,dz$$

$$= 2\int_{\theta=0}^{2\pi}\int_{z=0}^{3} (2x^2 - y^3)\,d\theta\,dz$$

$$= 2\int_{\theta=0}^{2\pi}\int_{z=0}^{3} (8\cos^2\theta - 8\sin^3\theta)\,d\theta\,dz$$

$$= 16\int_{\theta=0}^{2\pi}\int_{z=0}^{3} (\cos^2\theta - \sin^3\theta)\,d\theta\,dz$$

$$= 16\int_{\theta=0}^{2\pi} (\cos^2\theta - \sin^3\theta)(3)d\theta\,dz$$

$$= 48\int_{\theta=0}^{2\pi}\left(\frac{1}{2}(1 + \cos 2\theta) - \frac{1}{4}(3\sin\theta - \sin 3\theta\,)\right)d\theta\,dz$$

$$= 48\left[\frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) - \frac{1}{4}\left(-3\cos\theta + \frac{\cos 3\theta}{3}\right)\right]_0^{2\pi}$$

$$= 48\left[\frac{1}{2}(2\pi + 0) - \frac{1}{4}\left(-3(1) + \frac{1}{3}\right) - \frac{1}{2}(0) + \frac{1}{4}\left(-3(1) + \frac{1}{3}\right)\right]$$

$$= 48[\pi + 0] = 48\pi.$$

In S_2, $\vec{n} = -\vec{k}, z = 0$.

$\vec{A}.\vec{n} = (4x\vec{\imath} - 2y\vec{\jmath} + z^2\vec{k}).(-\vec{k}) = -z^2 = 0$.

$\therefore \iint_{S_2} (\vec{A}.\vec{n})\, dS = 0$.

In S_3, $\vec{n} = \vec{k}, z = 3$.

$\vec{A}.\vec{n} = (4x\vec{\imath} - 2y^2\vec{\jmath} + z^2\vec{k}).(\vec{k}) = z^2 = 9$.

$\therefore \iint_{S_3} (\vec{A}.\vec{n})\, dS = \iint_{S_3} 9\, dS$

$$= 9 \iint_{S_3} dS$$

$$= 9 \times (\text{area of the circle with radius } 2)$$

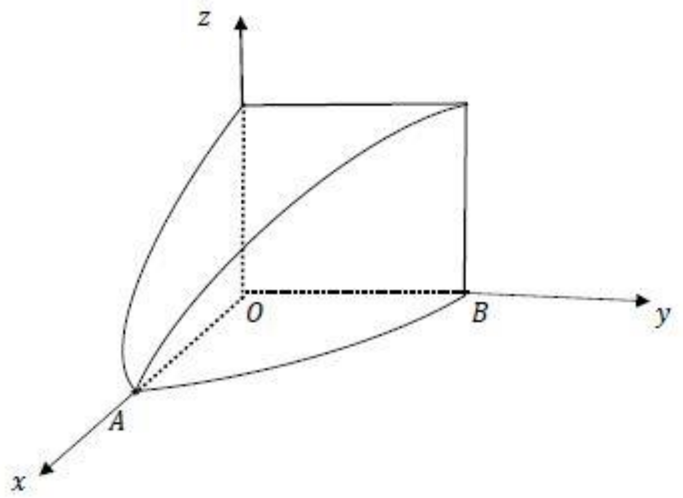
$$= 9(\pi \times 2^2)$$

$$= 36\pi$$

$$\therefore \iint_S (\vec{A}.\vec{n})\, dS = 48\pi + 0 + 36\pi = 84\pi.$$

Problem 11. ***Find the area of the curved surface of the region common to the cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$ contained in the first octant.***

Solution. Let the curved surface belonging to the cylinder $x^2 + y^2 = a^2$ be S_1 and the curved surface belonging to the cylinder $x^2 + z^2 = a^2$ be S_2. Project the surfaces S_1 on xy-plane and S_2 on xz-plane, respectively.



Now $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{\imath} + \frac{\partial\phi}{\partial y}\vec{\jmath} + \frac{\partial\phi}{\partial z}\vec{k} = 2x\vec{\imath} + 2z\vec{k}$$

$$|\nabla\phi| = \sqrt{4x^2 + 4z^2} = \sqrt{4(a^2)} = 2a$$

$$\therefore\ \vec{n} = \frac{1}{2a}\left(2x\vec{\imath} + 2z\vec{k}\right) = \frac{1}{a}\left(x\vec{\imath} + z\vec{k}\right)$$

$$\vec{n}.\vec{k} = \frac{1}{a}(z).$$

$$\therefore \text{Area of } S_1 = \iint_{S_1} dS_1 = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \frac{a}{z}\, dx\, dy$$

$$= a\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \frac{1}{\sqrt{a^2-x^2}} dydx$$

$$= a\int_{x=0}^{a} \frac{1}{\sqrt{a^2-x^2}} [y]_0^{\sqrt{a^2-x^2}} dx$$

$$= a\int_{x=0}^{a} \frac{1}{\sqrt{a^2-x^2}} \sqrt{a^2-x^2} dx$$

$$= a\int_{x=0}^{a} dx$$

$$= a(a) = a^2$$

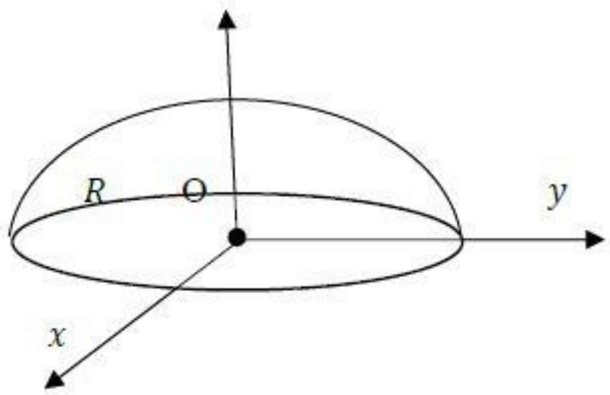
By symmetry, $\iint_{S_2} dS_2 = a^2$.

Required area $= a^2 + a^2 = 2a^2$.

Hemispherical Surface

Problem 12. ***Evaluate*** $\iint_S (\vec{A}.\vec{n})\, dS$ ***if*** $\vec{A} = x\vec{\imath} + y\vec{\jmath} - 2z\vec{k}$***and*** S ***is the surface of the sphere*** $x^2 + y^2 + z^2 = a^2$***above thexoy-plane.***

Solution. Given surface $\phi = x^2 + y^2 + z^2 - a^2$, $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$.



$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{\imath} + \frac{\partial\phi}{\partial y}\vec{\jmath} + \frac{\partial\phi}{\partial z}\vec{k} = 2x\vec{\imath} + 2y\vec{\jmath} + 2z\vec{k}$$

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2 + 4z^2} = \sqrt{4(a^2)} = 2a$$

$$\therefore\ \vec{n} = \frac{1}{2a}\left(2x\vec{i} + 2y\vec{j} + 2z\vec{k}\right) = \frac{1}{a}\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$\vec{n}.\vec{k} = \frac{1}{a}(z).$$

$$\vec{A}.\vec{n} = \left(x\vec{i} + y\vec{j} - 2z\vec{k}\right).\tfrac{1}{a}\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$= \frac{1}{a}(x^2 + y^2 - 2z^2)$$

$$= \frac{1}{a}\left(x^2 + y^2 - 2(a^2 - x^2 - y^2)\right)$$

$$= \frac{1}{a}(3x^2 + 3y^2 - 2a^2)$$

$$\therefore \iint_S \left(\vec{A}.\vec{n}\right) dS = \iint_{R_{xy}} \left(\vec{A}.\vec{n}\right) \frac{dxdy}{\left|\vec{n}.\vec{k}\right|}$$

$$= \iint_{R_{xy}} \frac{1}{a}(3x^2 + 3y^2 - 2a^2)\frac{a}{z} dxdy$$

$$= \iint_{R_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}}(3x^2 + 3y^2 - 2a^2)dxdy$$

Put $x = r\cos\theta, y = r\sin\theta$. Then $dxdy = rdrd\theta$.

r	0	a
t	0	2π

$$\therefore \iint_S \left(\vec{A}.\vec{n}\right) dS = \int_{r=0}^{a} \int_{\theta=0}^{2\pi} \frac{1}{\sqrt{a^2 - r^2}}(3r^2 - 2a^2)\, r\, dr\, d\theta$$

$$= \int_{r=0}^{a} \frac{1}{\sqrt{a^2 - r^2}}(3r^2 - 2a^2)\, r\, dr[\theta]_0^{2\pi}$$

$$= \int_{r=0}^{a} \frac{1}{\sqrt{a^2 - r^2}}(3r^2 - 2a^2)\, r\, dr(2\pi)$$

$$= 2\pi \int_{r=0}^{a} \frac{1}{\sqrt{a^2 - r^2}}(3r^2 - 2a^2)\, r\, dr$$

Put $a^2 - r^2 = t$. Then $-2r\, dr = dt$.

r	0	a
t	a^2	0

$$\therefore \iint_S \left(\vec{A}.\vec{n}\right) dS = 2\pi \int_{t=a^2}^{0} \frac{1}{\sqrt{t}}(3a^2 - 3t - 2a^2)(-dt)$$

$$= 2\pi \int_{t=a^2}^{0} \frac{1}{\sqrt{t}}(a^2 - 3t)\, dt$$

$$= 2\pi \int_{t=a^2}^{0} \frac{1}{\sqrt{t}}\left(a^2 t^{-1/2} - 3t^{1/2}\right) dt$$

$$= 2\pi \left[a^2 2t^{-1/2} - 3\left(\frac{2}{3}\right) t^{3/2}\right]_0^{a^2}$$

$$= 2\pi[2a^2 a \; - \; 2a^3]$$

$$= 2\pi[2a^3 \; - \; 2a^3]$$

$$= 0.$$

Problem 13. ***S*** ***is the surface of the hemisphere*** $\boldsymbol{x^2 + y^2 + z^2 = a^2, z \geq 0}$ ***and*** $\boldsymbol{\vec{A} = z\vec{k}}$**.** ***Show that*** $\boldsymbol{\iint_S (\vec{A}.\vec{n})\, dS = \iint_S (a^2 - x^2 - y^2)^{1/2}\, dxdy = \frac{2}{3}\pi a^3}$**.**

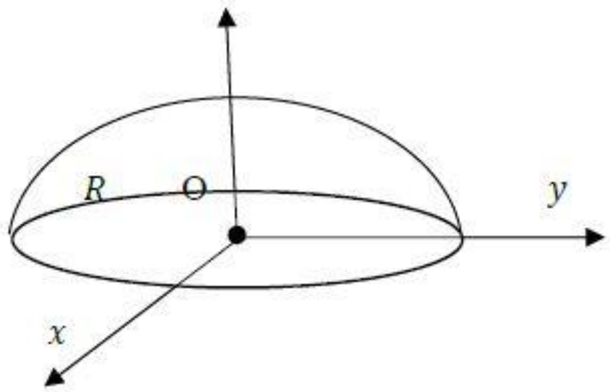
Solution. Given surface $\phi = x^2 + y^2 + z^2 - a^2, \vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{\imath} + \frac{\partial\phi}{\partial y}\vec{\jmath} + \frac{\partial\phi}{\partial z}\vec{k} = 2x\vec{\imath} + 2y\vec{\jmath} + 2z\vec{k}$$

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2 + 4z^2} = \sqrt{4(a^2)} = 2a$$

$$\therefore \; \vec{n} = \frac{1}{2a}\left(2x\vec{\imath} + 2y\vec{\jmath} + 2z\vec{k}\right) = \frac{1}{a}\left(x\vec{\imath} + y\vec{\jmath} + z\vec{k}\right)$$

$$\vec{n}.\vec{k} = \frac{1}{a}(z).$$



$$\vec{A}.\vec{n} = \left(z\vec{k}\right).\frac{1}{a}\left(x\vec{\imath} + y\vec{\jmath} + z\vec{k}\right)$$

$$= \frac{1}{a}(z^2)$$

$$\therefore \iint_S \left(\vec{A}.\vec{n}\right) dS = \iint_{R_{xy}} \left(\vec{A}.\vec{n}\right) \frac{dxdy}{|\vec{n}.\vec{k}|}$$

$$= \iint_{R_{xy}} \frac{1}{a}(z^2)\frac{a}{z}\, dxdy$$

$$= \iint_{R_{xy}} z\, dxdy$$

$$= \iint_{R_{xy}} (a^2 - x^2 - y^2)^{1/2}\, dxdy$$

Put $x = r\cos\theta, y = r\sin\theta$. Then $dxdy = rdrd\theta$.

r	0	a
θ	0	2π

$$\therefore \iint_S (\vec{A}.\vec{n})\, dS = \int_{r=0}^{a}\int_{\theta=0}^{2\pi} \sqrt{a^2 - r^2}\, r\, dr\, d\theta$$

$$= \int_{r=0}^{a} \sqrt{a^2 - r^2}\, r\, dr[\theta]_0^{2\pi}$$

$$= \int_{r=0}^{a} \sqrt{a^2 - r^2}\, r\, dr(2\pi)$$

$$= 2\pi \int_{r=0}^{a} \sqrt{a^2 - r^2}\, r\, dr$$

Put $a^2 - r^2 = t$. Then $-2r\, dr = dt$.

r	0	a
t	a^2	0

$$\therefore \iint_S (\vec{A}.\vec{n})\, dS = 2\pi \int_{t=a^2}^{0} \sqrt{t}(-dt)$$

$$= 2\pi \int_{t=0}^{a^2} (t^{1/2})\, dt$$

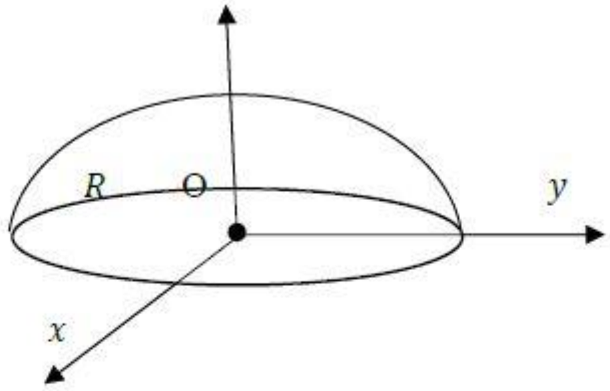
$$= 2\pi \left[\left(\frac{2}{3}\right) t^{3/2}\right]_0^{a^2}$$

$$= \frac{4}{3}\pi[a^3]$$

$$= \frac{4}{3}\pi a^3.$$

Problem 14. ***Find the area of the surface Sof the hemisphere*** $\boldsymbol{x^2 + y^2 + z^2 = a^2, z \geq 0}$.

Solution. Given surface $\phi = x^2 + y^2 + z^2 = a^2$,



$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2 + 4z^2} = \sqrt{4(a^2)} = 2a$$

$$\therefore\ \vec{n} = \frac{1}{2a}\left(2x\vec{i} + 2y\vec{j} + 2z\vec{k}\right) = \frac{1}{a}\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$\vec{n}.\vec{k} = \frac{1}{a}(z).$$

$$\therefore \text{Area of the surface} = \iint_S dS$$

$$= \iint_{R_{xy}} \frac{dxdy}{|\vec{n}.\vec{k}|}$$

$$= \iint_{R_{xy}} \frac{a}{z} dxdy$$

$$= a\iint_{R_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} dxdy$$

Put $x = r\cos\theta, y = r\sin\theta$. Then $dxdy = rdrd\theta$.

r	0	a
θ	0	2π

$$\therefore \text{Area of the surface} = \iint_S dS$$

$$= \int_{r=0}^{a}\int_{\theta=0}^{2\pi} \frac{1}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta$$

$$= \int_{r=0}^{a} \frac{1}{\sqrt{a^2 - r^2}}\, r\, dr[\theta]_0^{2\pi}$$

$$= \int_{r=0}^{a} \frac{1}{\sqrt{a^2 - r^2}}\, r\, dr(2\pi)$$

$$= 2\pi a \int_{r=0}^{a} \frac{1}{\sqrt{a^2 - r^2}}\, r\, dr$$

Put $a^2 - r^2 = t$. Then $-2r\, dr = dt$.

r	0	a
t	a^2	0

$$\therefore \text{Area of the surface} = \iint_S dS$$

$$= 2\pi a \int_{t=a^2}^{0} \frac{1}{\sqrt{t}}(-dt/2)$$

$$= \pi a \int_{t=0}^{a^2} \left(t^{-1/2}\right) dt$$

$$= \pi a \left[2t^{1/2}\right]_0^{a^2}$$

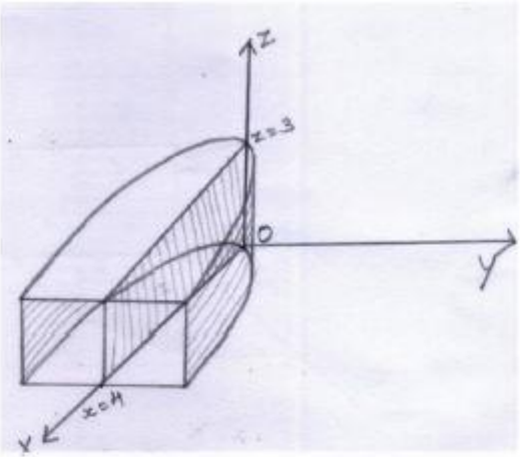
$$= \pi a[2a]$$

$$= 2\pi a.$$

Parabolic cylinder

Problem 15. ***Evaluate*** $\iint_S(\vec{A}.\vec{n})\, dS$ ***if*** $\vec{A} = y\vec{i} - x\vec{j} + z\vec{k}$ ***and*** S ***is the surface of the parabolic cylinder*** $y^2 - 4x = 0$ ***in the first octant bounded by the planes*** $x = 4$ ***and*** $z = 3$.

Solution. Let the projection R of S on the xoz-plane be the rectangle $OABC$.



Given $\vec{A} = y\vec{i} - x\vec{j} + z\vec{k}$ and ϕ is $y^2 - 4x = 0$.

Now, $\iint_S(\vec{A}.\vec{n})dS = \iint_{R_{xz}}(\vec{A}.\vec{n})\frac{dxdz}{|\vec{n}.\vec{j}|}$ where $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = -4\vec{i} + 2y\vec{j}$$

$$|\nabla\phi| = \sqrt{16+4y^2} = 2\sqrt{4+y^2}$$

$$\therefore\ \vec{n} = \frac{1}{2\sqrt{4+y^2}}(-4\vec{\imath}+2y\vec{\jmath}) = \frac{1}{\sqrt{4+y^2}}(-2\vec{\imath}+y\vec{\jmath})$$

Hence, $\vec{n}.\vec{\jmath} = \frac{y}{\sqrt{4+y^2}}$ and

$$\vec{A}.\vec{n} = \frac{1}{2\sqrt{4+y^2}}[-2y-xy] = \frac{y}{2\sqrt{4+y^2}}[-2-x]$$

$$\therefore \iint_S (\vec{A}.\vec{n})\,dS = \int_{x=0}^{4}\int_{z=0}^{3} \frac{y}{2\sqrt{4+y^2}}[-2-x]\frac{dxdz}{\frac{y}{\sqrt{4+y^2}}}$$

$$= \int_{x=0}^{4}\int_{z=0}^{3}(-2-x)\,dzdx$$

$$= \int_{x=0}^{4}(-2-x)\,[z]_0^3 dx$$

$$= 3\int_{x=0}^{4}(-2-x)\,dx$$

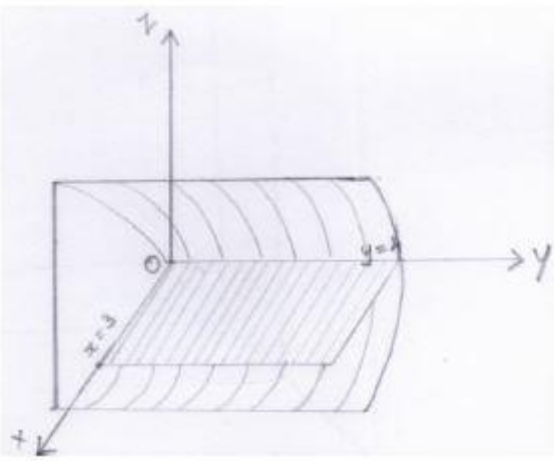
$$= 3\left[-2x-\frac{x^2}{2}\right]_0^4$$

$$= 3[-8-8]$$

$$= -48.$$

Problem 16. ***Evaluate*** $\iint_S(\vec{A}.\vec{n})\,dS$ ***if*** $\vec{A} = y\vec{\imath} - x\vec{\jmath} + z\vec{k}$ ***and S is the surface of the parabolic cylinder*** $z^2 - 4x = 0$ ***in the first octant bounded by the planes*** $x = 3$ ***and*** $y = 4$**.**

Solution. Let the projection R of S on the xoy-plane be the rectangle $OABC$. Now S is specified by $0 \le x \le 3$ and $0 \le y \le 4$.



Given $\vec{A} = y\vec{i} - x\vec{j} + z\vec{k}$and $\phi$is $z^2 - 4x = 0$.

Now, $\iint_S (\vec{A}.\vec{n})dS = \iint_{R_{xz}} (\vec{A}.\vec{n}) \frac{dxdz}{|\vec{n}.\vec{j}|}$ where $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = -4\vec{i} + 2z\vec{k}$$

$$|\nabla\phi| = \sqrt{16 + 2z^2} = 2\sqrt{4 + z^2}$$

$$\therefore\ \vec{n} = \frac{1}{2\sqrt{4 + z^2}}(-4\vec{i} + 2z\vec{j}) = \frac{1}{\sqrt{4 + z^2}}(-2\vec{i} + z\vec{k})$$

Hence, $\vec{n}.\vec{k} = \frac{z}{\sqrt{4+z^2}}$ and

$$\vec{A}.\vec{n} = \frac{1}{\sqrt{4 + z^2}}[-2y + z^2]$$

$$\therefore\ \iint_S (\vec{A}.\vec{n})\, dS = \int_{x=0}^{3}\int_{y=0}^{4} \frac{1}{\sqrt{4 + z^2}}[-2y + z^2] \frac{dxdz}{\frac{z}{\sqrt{4 + z^2}}}$$

$$= \int_{x=0}^{3}\int_{y=0}^{4} \frac{1}{z}(-2y + z^2)\, dydx$$

$$= \int_{x=0}^{3}\int_{y=0}^{4} \frac{1}{2\sqrt{x}}(-2y + 4x)\, dydx\ [z^2 = 4x]$$

$$= \int_{x=0}^{3} \frac{1}{2\sqrt{x}}[-y^2 + 4xy]_0^4\ dx$$

$$= \int_{x=0}^{3} \frac{1}{2\sqrt{x}}(-16 + 16x)\ dx$$

$$= 8\int_{x=0}^{3} \left[-x^{-\frac{1}{2}} + x^{\frac{1}{2}}\right] dx$$

$$= 8\left[-2x^{\frac{1}{2}} + \frac{2}{3}x^{3/2}\right]_0^3$$

$$= 8\left[-2\sqrt{x} + \frac{2}{3}x\sqrt{x}\right]_0^3$$

$$= 8[-2\sqrt{3} + 2\sqrt{3}]$$

$$= 0.$$

Problem 17. ***A surface Shas a projection Ron the xoy-plane. Show that the area of Sis,*** $\iint_S \left\{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}^{\frac{1}{2}} dxdy$ ***or*** $\iint_R \left[\frac{\left\{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}}{\left|\frac{\partial F}{\partial z}\right|}\right] dxdy$***according as the equation of S is*** $z = f(x,y)$***or*** $F(x,y,z) = 0$.

Solution: ***i***) $z = f(x,y)$.

$\therefore\ \phi = z - f(x,y) - 0.$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = \frac{\partial z}{\partial x}\vec{i} + \frac{\partial z}{\partial y}\vec{j} + 1\,\vec{k}$$

$$|\nabla\phi| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

$$\therefore\ \vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\frac{\partial z}{\partial x}\vec{i} + \frac{\partial z}{\partial y}\vec{j} + 1\,\vec{k}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$$

$$\text{Hence } \vec{n}.\vec{k} = \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}.$$

$$\therefore \iint_S dS = \iint_R \frac{dxdy}{|\vec{n}.\vec{k}|}$$

$$= \iint_R \left\{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}^{1/2} dxdy$$

ii) $\phi = F(x,y,z) = 0.$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k}$$

$$|\nabla\phi| = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}$$

$$\therefore\ \vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}$$

$$\text{Hence } \vec{n}.\vec{k} = \frac{\frac{\partial F}{\partial z}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}.$$

$$\therefore \iint_S dS = \iint_R \frac{dxdy}{|\vec{n}.\vec{k}|}$$

$$= \iint_R \left[\frac{\left\{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}}{\left|\frac{\partial F}{\partial z}\right|}\right] dxdy$$

4.2. Volume Integral

The volume integral is denoted by $\iiint_V (function)\, dV$.

If the vector is $\vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$, then the volume integral is

$$\iiint_V \vec{f}\ dV = \vec{i}\iiint_V f_1\ dV + \vec{j}\iiint_V f_2\ dV + \vec{k}\iiint_V f_3\ dV$$

Cylindrical co-ordinates

The relation between x, y, z and r, θ, z and that between $dx\ dy\ dz$ and $dr\ d\theta\ dz$ are $x = r\cos\theta, y = r\sin\theta, z = z$ and $dx\ dy\ dz = (r\ dr\ d\theta)\ dz$.

Spherical co-ordinates

The relation between $x, y, z$and r, θ, φ and that between $dx\ dy\ dz$ and $dr\ d\theta\ d\varphi$ are $x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta$ and $dx\ dy\ dz = (r2\sin\theta)\ dr\ d\theta\ d\varphi$.

Example 1. ***Evaluate*** $\iiint_V \nabla.\vec{F}\,dV$***if*** $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$***and if V is the volume of the regionenclosed by the cube*** $0 \le x, y, z \le 1$.

Solution.

$$\nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(x^2\vec{i} + y^2\vec{j} + z^2\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$

$$\therefore \nabla.\vec{F} = 2(x + y + z)$$

$$\iiint_V \nabla.\vec{F}\,dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} 2(x + y + z)\,dx\,dy\,dz$$

$$= 2\int_{x=0}^{1}\int_{y=0}^{1}\left[xz + yz + \frac{z^2}{2}\right]_1^0 dy\,dx$$

$$= 2\int_{x=0}^{1}\int_{y=0}^{1}\left(x + y + \frac{1}{2}\right) dy\,dx$$

$$= 2\int_{x=0}^{1}\left[xy + \frac{y^2}{2} + \frac{y}{2}\right]_0^1 dx$$

$$= 2\int_{x=0}^{1}\left(x + \frac{1}{2} + \frac{1}{2}\right) dx$$

$$= 2\int_{x=0}^{1}(x + 1)\,dx$$

$$= 2\left[\frac{x^2}{2} + x\right]_0^1$$

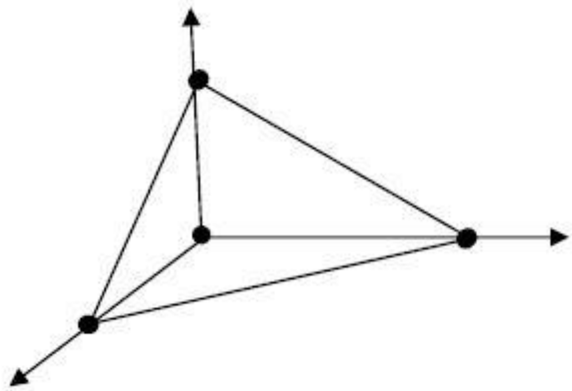
$$= 2\left[\frac{1}{2} + 1\right]$$

$$= 2 \times \frac{3}{2}$$

$$= 3.$$

Example 2. $\iiint_V 45x^2y\,dV$, *where V is the region bounded by the planes* $x = 0, y = 0, z = 0, 4x + 2y + z = 8.$

Solution:



$$\iiint_V 45x^2y\,dV = \int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-4x-2y} 45x^2y\,dz\,dy\,dx$$

$$= 45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2y\,[z]_0^{8-4x-2y} dy\,dx$$

$$= 45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2y\,(8-4x-2y)dy\,dx$$

$$= 45\int_{x=0}^{2}\int_{y=0}^{4-2x} [8x^2y - 4x^3y - 2x^2y^2]\,dy\,dx$$

$$= 45\int_{x=0}^{2}\int_{y=0}^{4-2x} \left[\frac{8x^2y^2}{2} - \frac{4x^3y^2}{2} - \frac{2x^2y^3}{3}\right]_0^{4-2x} dx$$

$$= 45\int_{x=0}^{2}\left[4x^2(4-2x)^2 - 2x^3(4-2x)^2 - \frac{2x^3}{3}(4-2x)^3\right]dx$$

$$= \frac{45}{3}\int_{x=0}^{2} [12x^2(16-16x+4x^2) - 6x^3(16-16x+4x^2)$$

$$-2x^2(64 - 96x + 48x^2 - 8x^3)]dx$$

$$= 15\int_{x=0}^{2} [192x^2 - 192x^3 + 48x^4 - 96x^3 + 96x^4 - 24x^5 - 128x^2$$

$$+192x^3 - 96x^4 + 16x^5]dx$$

$$= 15\int_{x=0}^{2} [-8x^5 + 48x^4 - 96x^3 + 64x^2]\, dx$$

$$= 15\left[-8\frac{x^6}{6} + 48\frac{x^5}{5} - 96\frac{x^4}{4} + 64\frac{x^3}{3}\right]_0^2$$

$$= 15\left[\frac{-4 \times 64}{3} + \frac{48 \times 32}{5} - 24 \times 16 + \frac{64 \times 4}{3}\right]$$

$$= 15\left[\frac{-1280 + 4608 - 5760 + 2560}{15}\right]$$

$$= 7168 - 7040$$

$$= 128.$$

$$\iiint_V 45\mathrm{x}^2 y\, dV = 128.$$

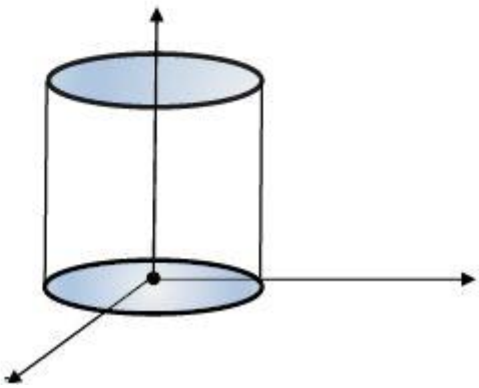
Example 3. ***Evaluate*** $\iiint_V \nabla.\vec{F}\, dV$***, where*** $\vec{F} = 2xz\vec{i} - x\vec{j} + y^2\vec{k}$ ***and V is the volume of the regionenclosed by the cylinder*** $x^2 + y^2 = a^2$ ***between the planes*** $z = 0, z = c$***.***

Solution. Given cylinder is $x^2 + y^2 = a^2, z = 0, z = c$.

Then $x = r\cos\theta, y = r\sin\theta, z = z$

and $dxdydz = rdrd\theta dz$.

Also $r: 0 \to a;\ \theta: 0 \to 2\pi$ and $z: 0 \to c$.



$$\iiint_V \vec{F}\, dV = \iiint_V \left(2xz\vec{i} - x\vec{j} + y^2\vec{k}\right) dV$$

$$= 2\vec{i}\iiint_V xz\, dV - \vec{j}\iiint_V x\, dV + \vec{k}\iiint_V y^2\, dV$$

$$\iiint_V \vec{F}\, dV = I - 1 - I_2 + I_3 \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

$$I_1 = 2\vec{i}\iiint_V xz\,dV$$

$$= 2\vec{i}\int_{r=0}^{a}\int_{\theta=0}^{2\pi}\int_{z=0}^{c} r\cos\theta\, z(r\,dz\,d\theta dr)$$

$$= 2\vec{i}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2\cos\theta\left(\frac{z^2}{2}\right)_0^c d\theta dr$$

$$= 2\vec{i}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2\cos\theta\left(\frac{c^2}{2}\right) d\theta dr$$

$$= c^2\vec{i}\int_{r=0}^{a} r^2\,[\sin\theta]_0^{2\pi} dr$$

$$= 0.$$

$$\therefore I_1 = 0. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$I_2 = \vec{j}\iiint_V x\,dV$$

$$= \vec{j}\int_{r=0}^{a}\int_{\theta=0}^{2\pi}\int_{z=0}^{c} r\cos\theta\, z(r\,dz\,d\theta dr)$$

$$= \vec{j}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2\cos\theta\,[z]_0^c d\theta dr$$

$$= c\vec{j}\int_{r=0}^{a} r^2\,[\sin\theta]_0^{2\pi} dr$$

$$= 0.$$

$$\therefore I_2 = 0. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$I_3 = \vec{k}\iiint_V y^2\,dV$$

$$= \vec{k}\int_{r=0}^{a}\int_{\theta=0}^{2\pi}\int_{z=0}^{c} r^2\sin^2\theta\, z(r\,dz\,d\theta dr)$$

$$= \vec{k}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^3\sin^2\theta\,[z]_0^c d\theta dr$$

$$= c\vec{k}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^3\left[\frac{1-\cos 2\theta}{2}\right] d\theta dr$$

$$= c\vec{k}\int_{r=0}^{a} r^3\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_0^{2\pi} dr$$

$$= c\vec{k}\int_{r=0}^{a} r^3\left(\frac{1}{2}\times 2\pi\right) dr$$

$$= \pi c\vec{k}\left[\frac{r^4}{4}\right]_0^a$$

$$= \frac{a^4\pi c\vec{k}}{4}.$$

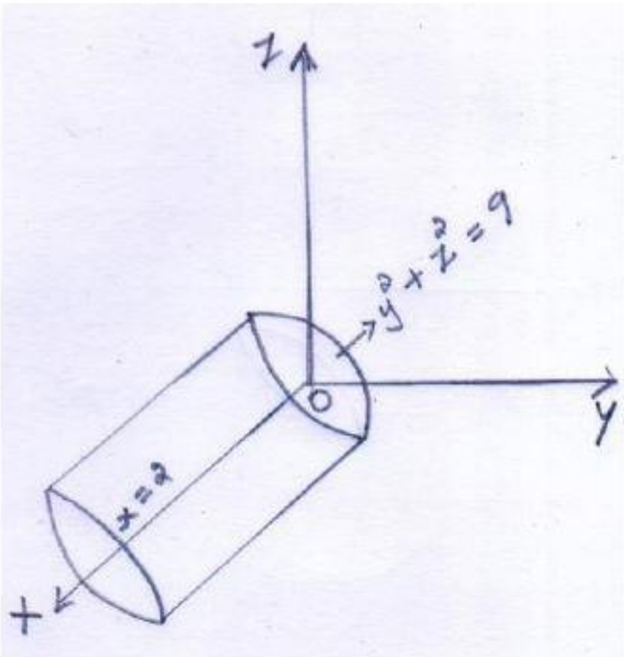
$$\therefore I_1 = \frac{a^4\pi c\vec{k}}{4}. \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (4)$$

Substituting (2), (3) and (4) in (1) we get,

$$\iiint_V \vec{F}\, dV = 0 - 0 + \frac{a^4\pi c\vec{k}}{4}$$

$$= \frac{a^4\pi c\vec{k}}{4}.$$

Example 4. ***Evaluate*** $\iiint_V \nabla.\vec{A}\, dV$ ***if*** $\vec{A} = 2x^2\vec{\imath} - y^2\vec{\jmath} + 4xz^2\vec{k}$ ***and V is the region in the first octantbounded by the cylinder*** $y^2 + z^2 = 9$ ***andtheplanex*** $= 2$.



$$\nabla.\vec{A} = \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2)$$

$$= 4xy - 2y + 8xz$$

Here, x varies from $0 \to 2$, y varies from $0 \to 3$ and z varies from $0 \to \sqrt{9} - y2$.

The cylindrical co-ordinates for (x, y, z) is (x, r, θ) where $x = x$, $y = rcos\theta$, $z = rsin\theta$.

Now, x varies from $0 \to 2$, r varies from $0 \to 3$ and θ varies from $0 \to \frac{\pi}{2}$.

Also $dx\, dy\, dz = r\, dx\, dr\, d\theta$.

$$\iiint_V \nabla.\vec{A}\, dV = \int_{x=0}^{2}\int_{r=0}^{3}\int_{\theta=0}^{\frac{\pi}{2}} (4xy - 2y + 8xz)\, r\, d\theta\, dr\, dx$$

$$= \int_{x=0}^{2}\int_{r=0}^{3}\int_{\theta=0}^{\frac{\pi}{2}} r(4x\, r\cos\theta - 2r\cos\theta + 8xr\sin\theta)\, d\theta\, dr\, dx$$

$$= \int_{x=0}^{2}\int_{r=0}^{3}\int_{\theta=0}^{\frac{\pi}{2}} (4x\,r^2\cos\theta \;-\; 2r^2\cos\theta \;+\; 8x\,r^2\sin\theta)\;d\theta\,dr\,dx$$

$$= \int_{x=0}^{2}\int_{r=0}^{3} [(4x\,r^2\sin\theta \;-\; 2r^2\sin\theta - \;8x\,r^2\cos\theta)]_0^{\frac{\pi}{2}}\;\;dr\,dx$$

$$= \int_{x=0}^{2}\int_{r=0}^{3} \left[\left(4x\,r^2 - \;2r^2 - \;8x\,r^2(0) - (-8xr^2)\right)\right]_0^{\frac{\pi}{2}}\;\;dr\,dx$$

$$= \int_{x=0}^{2}\int_{r=0}^{3} (4x\,r^2 - \;2r^2 + 8xr^2)\;\;dx\,dr$$

$$= \int_{x=0}^{2}\left[\frac{4x\,r^3}{3} - \frac{2r^3}{3} + \frac{8xr^3}{3}\right]_0^3 dx$$

$$= \int_{x=0}^{2}\left[\frac{12x\,r^3}{3} - \frac{2r^3}{3}\right]_0^3 dx$$

$$= \int_{x=0}^{2}\left(4x \times 27 - \frac{2}{3}\times 27\right)dx$$

$$= \int_{x=0}^{2}(108x - 18)\,dx$$

$$= \left[\frac{108x^2}{2} - 18x\right]_0^2$$

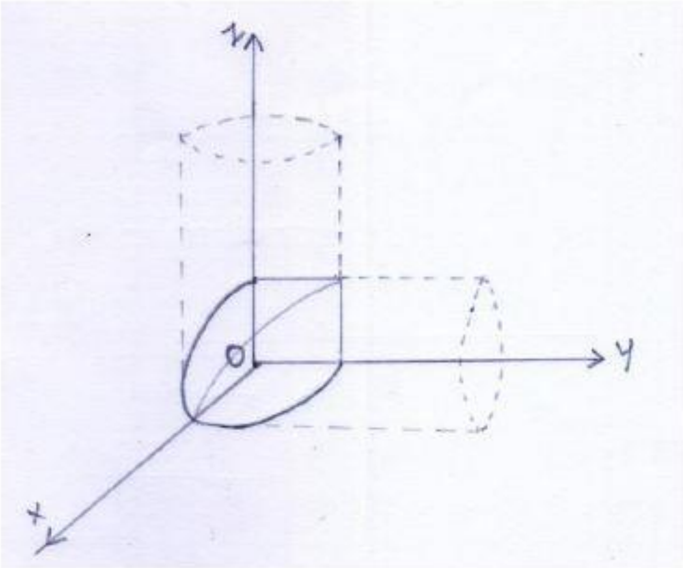
$$= 54 \times 4 - 18 \times 2$$

$$\iiint_V \nabla.\vec{A}\,dV = 216 - 36$$

$$= 180.$$

Example 5. ***Evaluate the following integral over the region common to the cylinders*** $x^2 + y^2 = a^2$***and***$x^2 + z^2 = a^2$***contained in the first octant if*** $\vec{A} = xy\vec{\imath} - 3y^2z\vec{\jmath}$, $\iiint \nabla \times \vec{A}dV = ?$

Solution:



Given, $\vec{A} = xy\vec{\imath} - 3y^2z\vec{\jmath}$.

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -3y^2z & 0 \end{vmatrix}$$

$$= \vec{\imath}[0 + 3y^2] - \vec{\jmath}[0 - 0] + \vec{k}[0 - x]$$

$$= 3y^2\vec{\imath} - x\vec{k}$$

Given $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

$\therefore\ x: 0 \to a, y: 0 \to \sqrt{a^2 - x^2} and z: 0 \to \sqrt{a^2 - x^2}$.

$$\iiint_V \nabla \times \vec{A}\, dV = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2}} \left(3y^2\vec{\imath} - x\vec{k}\right) dz\, dy\, dx$$

$$= \vec{\imath} \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2}} 3y^2\, dz\, dy\, dx$$

$$-\vec{k} \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2}} x\, dz\, dy\, dx$$

$$= \vec{\imath} \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} 3y^2\, [z]_{z=0}^{\sqrt{a^2-x^2}}\, dy\, dx$$

$$-\vec{k} \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} x\, [z]_{z=0}^{\sqrt{a^2-x^2}}\, dy\, dx$$

$$= \vec{\imath} \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} 3y^2 \sqrt{a^2 - x^2}\, dy\, dx$$

$$-\vec{k} \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} x \sqrt{a^2 - x^2}\, dy\, dx$$

$$= \vec{i}\int_{x=0}^{a} 3\sqrt{a^2-x^2}\left[\frac{y^3}{3}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$-\vec{k}\int_{x=0}^{a} x\sqrt{a^2-x^2}[y]_0^{\sqrt{a^2-x^2}} dx$$

$$= \vec{i}\int_{x=0}^{a} \sqrt{a^2-x^2}\sqrt{a^2-x^2}(a^2-x^2)\ dx$$

$$-\vec{k}\int_{x=0}^{a} x\sqrt{a^2-x^2}\sqrt{a^2-x^2}\, dx$$

$$= \vec{i}\int_{x=0}^{a} (a^2-x^2)(a^2-x^2)\ dx - \vec{k}\int_{x=0}^{a} x\,(a^2-x^2)\, dx$$

$$= \vec{i}\int_{x=0}^{a} (a^4-2a^2x^2+x^4)\ dx - \vec{k}\int_{x=0}^{a} (a^2x-x^3)\ dx$$

$$= \vec{i}\left[a^4x-\frac{2a^2x^3}{3}+\frac{x^5}{5}\right]_0^a - \vec{k}\left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right]_0^a$$

$$= \vec{i}\left[a^5-\frac{2a^5}{3}+\frac{a^5}{5}\right] - \vec{k}\left[\frac{a^4}{2}-\frac{a^4}{4}\right]$$

$$= \vec{i}\left[\frac{15a^5-10a^5+3a^5}{15}\right] - \vec{k}\left[\frac{2a^4-a^4}{4}\right]$$

$$= \vec{i}\left[\frac{18a^5-10a^5}{15}\right] - \vec{k}\left[\frac{a^4}{4}\right]$$

$$= \vec{i}\left[\frac{8a^5}{15}\right] - \vec{k}\left[\frac{a^4}{4}\right]$$

$$\iiint_V \nabla\times\vec{A}\,dV = \vec{i}\left[\frac{8a^5}{15}\right] - \vec{k}\left[\frac{a^4}{4}\right].$$

UNIT V

5.1 GAUSS' DIVERGENCE THEOREM

If V is the volume of a closed surface S and A, a vector point function with continuous derivatives in V, then $\iint_S A \cdot dS = \iiint_V \nabla \cdot A \, dV$.

Problems

Problem 1: ***Show that*** $\iint_S \vec{r} \cdot n \, dS = 4\pi a^3$ ***if S is the surface of the sphere*** $x^2 + y^2 + z^2 = a^2$.

Solution. By Gauss' divergence $\iint_S \vec{r} \cdot dS = \iiint_V \nabla \cdot r \, dV$ .... (1)

$$\nabla \cdot \vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3.$$

$$(1) \Rightarrow \iiint_V \nabla \cdot \vec{r} \, dV = \iiint_V 3 \, dV$$

$$= 3\frac{4}{3}\pi a^3 = 4\pi a^3$$

$$Hence\ proved.$$

Problem 2: ***Show that the volume V of the region enclosed by the surface S is*** $\frac{1}{3}\iint_S \vec{r} \cdot dS$.

Solution. By Gauss' divergence $\iint_S \vec{r} \cdot dS = \iiint_V \nabla \cdot \vec{r} \, dV$ .... (1)

$$\frac{1}{3}\iint_S \vec{r} \cdot dS = \frac{1}{3}\iiint_V \nabla \cdot \vec{r} \, dV$$

$$= \frac{1}{3}\iiint_V 3 \, dV$$

$$= \frac{1}{3} 3 \iiint_V dV = V.$$

Thus, V=$\frac{1}{3}\iint_S \vec{r} \cdot dS$

Hence proved.

Problem 3: Evaluate $\iint_S \vec{r} \cdot dS$ ***where (i) S is the sphere*** $x^2 + y^2 + z^2 = 9$

(ii) S is the cube bounded by $\boldsymbol{x=-1, x=1, y=-1, y=1, z=-1, z=1}$.

Solution : (i) By Gauss' divergence $\iint_S \vec{r}\cdot dS = \iiint_V \nabla\cdot\vec{r}\, dV$ ……………… (1)

$$\nabla\cdot\vec{r} = \left(\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(x\vec{\imath} + y\vec{\jmath} + z\vec{k}\right)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3.$$

$$(1) \Rightarrow \iiint_V \nabla\cdot\vec{r}\, dV = \iiint_V 3\, dV$$

$$= 3\frac{4}{3}\pi 3^3 = 108\pi.$$

(ii) By Gauss' divergence $\iint_S \vec{r}\cdot dS = \iiint_V \nabla\cdot\vec{r}\, dV$ ……………… (1)

$$(1) \Rightarrow \iiint_V \nabla\cdot\vec{r}\, dV = \iiint_V 3\, dV$$

$$= 3\int_{x=-1}^{1}\int_{y=-1}^{1}\int_{z=-1}^{1} dzdydx$$

$$= 3\int_{x=-1}^{1}\int_{y=-1}^{1} [z]_{-1}^{1} dydx$$

$$= 3\int_{x=-1}^{1}\int_{y=-1}^{1} (1+1) dydx$$

$$= 3\times 2\int_{x=-1}^{1} [y]_{-1}^{1} dx$$

$$= 6\int_{x=-1}^{1} (1+1) dx$$

$$= 6\times 2[x]_{-1}^{1}$$

$$= 12\times 2 = 24.$$

Problem 4: Show, for a closed surface S enclosing a region of volume V, that $\boldsymbol{\iint_S (ax\vec{\imath} + by\vec{\jmath} + cz\vec{k})\cdot\vec{n}dS = (a+b+c)V}$.

Problem 5: If $\boldsymbol{\vec{F} = x^3\vec{\imath} + y^3\vec{\jmath} + z^3\vec{k}}$ ***and S is the surface of the sphere*** $\boldsymbol{x^2+y^2+z^2 = a^2}$, ***show that*** $\boldsymbol{\iint_S \vec{F}\cdot\vec{n}dS = \frac{12}{5}\pi a^5}$.

Solution : By Gauss' divergence $\iint_S \vec{F}\cdot dS = \iiint_V \nabla\cdot\vec{F}\, dV$

$$\nabla\cdot\vec{F} = 3(x^2+y^2+z^2)$$

Using spherical polar co-ordinates, $x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi,\ z = r\cos\theta$

$$dxdydz = r^2\sin\theta dr d\theta d\varphi.$$

$$r \text{ varies from } 0 \text{ to } a;\ \theta: 0 \to \pi;\ \varphi: 0 \to 2\pi$$

$$\iiint_V \nabla\cdot\vec{F}\,dV = \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} 3(r^2\sin^2\theta\cos^2\varphi + r^2\sin^2\theta\sin^2\varphi + r^2\cos^2\theta)r^2\sin\theta d\varphi d\theta dr$$

$$= \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} 3r^4(\sin^2\theta(\cos^2\varphi + \sin^2\varphi) + \cos^2\theta)\sin\theta d\varphi d\theta dr$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} r^4(\sin^2\theta + \cos^2\theta)\sin\theta d\varphi d\theta dr$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} r^4\sin\theta d\varphi d\theta dr$$

$$= 3\left[\left(\frac{r^5}{5}\right)_0^a (-\cos\theta)_0^{\pi}(\varphi)_0^{2\pi}\right]$$

$$= \frac{12}{5}\pi a^5.$$

Problem 6: Evaluate $\iint_S \vec{F}\cdot\vec{n}dS$ ***if*** $\vec{F} = x\vec{i} + y\vec{j} - 2z\vec{k}$ ***and S is the surface of the upper hemisphere*** $x^2 + y^2 + z^2 = a^2$. [Ans. 0]

Problem 7: Evaluate $\iint_S \vec{A}\cdot\vec{n}dS$ ***if*** $\vec{A} = 2xy\vec{i} + yz^2\vec{j} + xz\vec{k}$ ***and S is the surface of the parallelepiped formed by the planes x=0, x=2, y=0, y=2, z=0, z=3.*** [Ans. 30]

Problem 8: Show that $\iint_S \vec{A}\cdot\vec{n}dS = \frac{3}{2}$ ***if*** $\vec{A} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ ***and S is the surface of the cube bounded by the planes x=0, x=1, y=0, y=1, z=0, z=1.***

Problem 9: Evaluate $\iint_S \vec{A}\cdot\vec{n}dS$***, where*** $\vec{A} = xz\vec{i} - yz\vec{j} + 2z^2\vec{k}$ ***and S is the surface of the region bounded by the following surfaces:***

(i) ***x=0, x=1; y=0, y=2; z=0, z=3.*** [Ans. 36]

(ii) ***x=0, y=0, z=0, 4x+2y+z=8.*** [Ans. 256/3]

(iii) ***x=0, y=0, z=0, z=2,*** $x^2 + y^2 - 9 = 0$. [Ans. 18π]

(iv) ***x=0, y=0, z=0,*** $x^2 + y^2 = a^2, x^2 + z^2 = a^2$. [Ans. $\frac{3\pi a^4}{8}$]

(v) ***x=0, y=0, y=3, z=25,z=*** x^2. [Ans. 15000].

Verification of Divergence Theorem

Problem 10: Verify Gauss divergence theorem for the vector $\vec{A} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ ***taken over the parallelepiped defined by*** $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.

Solution : By Gauss' divergence $\iint_S \vec{r} \cdot dS = \iiint_V \nabla \cdot \vec{r}\, dV$

Suppose the faces whose equations are $x = 0, x = a, y = 0, y = b, z = 0, z = c$ are respectively named $S_1, S_2, S_3, S_4, S_5, S_6$ and $\vec{n}$ denotes the unit vector normal to them.

On $S_1(YZ - plane)$, take $x = 0, \vec{n} = -\vec{i}$

$$\vec{A} \cdot \vec{n} = -x^2 + yz = yz.$$

$$\iint_{S_1} \vec{A} \cdot \vec{n} dS = \int_0^b \int_0^c yz\, dydz$$

$$= \int_0^b y \left[\frac{z^2}{2}\right]_0^c dy$$

$$= \int_0^b y \frac{c^2}{2} dy$$

$$= \frac{c^2}{2}\left[\frac{y^2}{2}\right]_0^b$$

$$= \frac{c^2}{2}\frac{b^2}{2}$$

$$= \frac{(bc)^2}{4}$$

On $S_2(YZ - plane)$, take $x = a, \vec{n} = \vec{i}$

$$\vec{A} \cdot \vec{n} = x^2 - yz = a^2 - yz.$$

$$\iint_{S_2} \vec{A} \cdot \vec{n} dS = \int_0^b \int_0^c (a^2 - yz)\, dydz$$

$$= \int_0^b \left[a^2 z - \frac{yz^2}{2}\right]_0^c dy$$

$$= \int_0^b \left[a^2 c - \frac{yc^2}{2}\right] dy$$

$$= a^2 cb - \frac{c^2 b^2}{4}$$

On $S_3 (XZ - plane)$, take $y = 0, \vec{n} = -\vec{j}$

$$\vec{A} \cdot \vec{n} = zx$$

$$\iint_{S_3} \vec{A} \cdot \vec{n} dS = \int_0^a \int_0^c zx \, dz dx$$

$$= \int_0^a x \left[\frac{z^2}{2}\right]_0^c dx$$

$$= \int_0^a x \frac{c^2}{2} dx$$

$$= \frac{c^2}{2} \left[\frac{x^2}{2}\right]_0^a$$

$$= \frac{c^2}{2} \frac{a^2}{2}$$

$$= \frac{(ac)^2}{4}$$

On $S_4 (XZ - plane)$, take $y = b, \vec{n} = \vec{j}$

$$\vec{A} \cdot \vec{n} = y^2 - zx = b^2 - zx.$$

$$\iint_{S_4} \vec{A} \cdot \vec{n} dS = \int_0^a \int_0^c (b^2 - zx) \, dz dx$$

$$= \int_0^a \left[b^2 z - \frac{xz^2}{2}\right]_0^c dx$$

$$= \int_0^a \left[b^2 c - \frac{xc^2}{2}\right] dx$$

$$= b^2ac - \frac{c^2a^2}{4}$$

On $S_5(XY - plane)$, take $z = 0, \vec{n} = -\vec{k}$

$$\vec{A} \cdot \vec{n} = xy$$

$$\iint_{S_5} \vec{A} \cdot \vec{n} dS = \int_0^a \int_0^b xy\, dxdy$$

$$= \int_0^a x \left[\frac{y^2}{2}\right]_0^b dx$$

$$= \int_0^a x \frac{b^2}{2} dx$$

$$= \frac{b^2}{2}\left[\frac{x^2}{2}\right]_0^a$$

$$= \frac{b^2}{2}\frac{a^2}{2}$$

$$= \frac{(ab)^2}{4}$$

On $S_6(XY - plane)$, take $y = b, \vec{n} = \vec{k}$

$$\vec{A} \cdot \vec{n} = c^2 - xy.$$

$$\iint_{S_6} \vec{A} \cdot \vec{n} dS = \int_0^a \int_0^b (c^2 - xy)\, dxdy$$

$$= \int_0^a \left[c^2y - \frac{xy^2}{2}\right]_0^b dx$$

$$= \int_0^a \left[c^2b - \frac{xb^2}{2}\right] dx$$

$$= c^2ab - \frac{a^2b^2}{4}$$

$$\iint_S \vec{A}\cdot\vec{n}dS = \iint_{S_1} \vec{A}\cdot\vec{n}dS + \iint_{S_2} \vec{A}\cdot\vec{n}dS + \iint_{S_3} \vec{A}\cdot\vec{n}dS + \iint_{S_4} \vec{A}\cdot\vec{n}dS + \iint_{S_5} \vec{A}\cdot\vec{n}dS + \iint_{S_6} \vec{A}\cdot\vec{n}dS$$

$$= \frac{(bc)^2}{4} + a^2cb - \frac{c^2b^2}{4} + \frac{(ac)^2}{4} + b^2ac - \frac{c^2a^2}{4} + \frac{(ab)^2}{4} + c^2ab - \frac{a^2b^2}{4}$$

$$= abc(a+b+c) \ldots\ldots\ldots\ldots\ldots(1).$$

Now, $\nabla\cdot\vec{A} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left((x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right)$

$$= 2(x+y+z).$$

$$\iiint_V \nabla\cdot\vec{A}\,dV = \int_0^a\int_0^b\int_0^c 2(x+y+z)dzdydx$$

$$= \int_0^a\int_0^b 2\left[xz + yz + \frac{z^2}{2}\right]_0^c dydx$$

$$= 2\int_0^a\int_0^b \left[xc + yc + \frac{c^2}{2}\right] dydx$$

$$= 2\int_0^a \left[xcy + \frac{y^2}{2}c + \frac{c^2y}{2}\right]_0^b dx$$

$$= 2\int_0^a \left[xcb + \frac{b^2}{2}c + \frac{c^2b}{2}\right] dx$$

$$= 2\left[\frac{x^2}{2}cb + \frac{b^2}{2}cx + \frac{c^2b}{2}x\right]_0^a$$

$$= \frac{2[a^2cb + ab^2c + abc^2]}{2}$$

$$= abc(a+b+c) \ldots\ldots\ldots\ldots\ldots(2).$$

From (1) & (2) Gauss divergence theorem is verified.

Problem 11: Verify the divergence theorem for $\vec{A} = (x+y)\vec{i} + x\vec{j} + z\vec{k}$ taken over the region V of the cube bounded by the planes x=0, x=1, y=0, y=1, z=0, z=1.

$$\left[Ans. \iint_S \vec{r} \cdot dS = \iiint_V \nabla \cdot \vec{r}\, dV = 2\right]$$

Problem 12: Verify the divergence theorem for $\vec{A} = 4x\vec{\imath} - 2y^2\vec{\jmath} + z^2\vec{k}$ taken over the cylindrical region bounded by the surfaces $x^2 + y^2 = 4, z = 0, z = 3$.

[Ans. $\iint_S \vec{r} \cdot dS = \iiint_V \nabla \cdot \vec{r}\, dV = 84\pi$]

Problem 13: Verify the divergence theorem for $\vec{A} = yz\vec{\imath} + 2y^2\vec{\jmath} + xz^2\vec{k}$ taken over the region bounded y x=0, y=, z=0, z=2 and $x^2 + y^2 = 9$.

[Ans. $\iint_S \vec{r} \cdot dS = \iiint_V \nabla \cdot \vec{r}\, dV = 108$].

5.2 Green's Theorem

Green's theorem in plane

If C is a simple closed curve in the xy plane bounding an area R and M(x,y) and N(x,y) are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

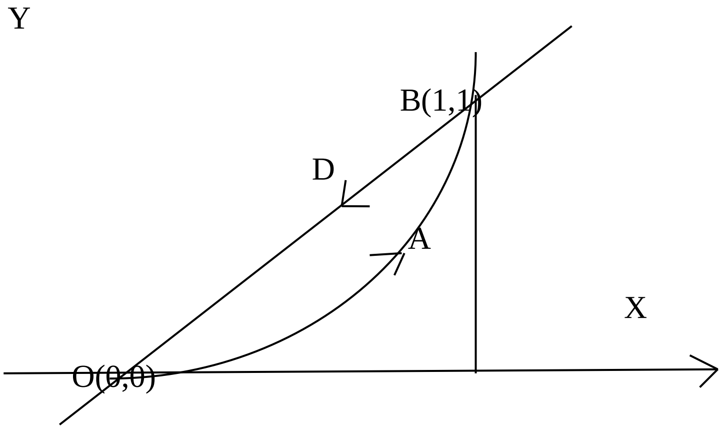
Problem 1 : Verify Green's theorem in plane for the integral $\int_C (xy + y^2)dx + x^2dy$, where C is the curve enclosing the region R bounded by the parabola $y = x^2$ and the line $y = x$.

Solution : Given $y = x^2$ and $y = x$.

Therefore, $x = x^2 => x^2 - x = 0 => x(x-1) = 0 => x = 0 \; or \; x = 1$

When x=0, y=0 & when x=1, y=1

Thus the parabola and the line intersect at (0,0) and (1,1).



In the figure OABDO, the curve C consists of the parabolic arc OAB and the line segment BDO.

The parametric equations of OAB are $x = t, y = t^2$ where t varies from 0 to 1.

Here, $M = xy + y^2$ & $N = x^2$

$$\therefore M = t \times t^2 + t^4 = t^3 + t^4 \;\&\; N = t^2$$

$$dx = dt \;\&\; dy = 2tdt$$

$$\int_C (xy + y^2)dx + x^2dy = \int_0^1 (t^3 + t^4)dt + t^2 2tdt = \int_0^1 (t^3 + t^4 + 2t^3)dt$$

$$= \left[\frac{t^4}{4} + \frac{t^5}{5} + \frac{2t^4}{4}\right]_0^1 = \frac{1}{4} + \frac{1}{5} + \frac{1}{2} - 0 = \frac{19}{20}$$

The parametric equations of BDO are $x = t, y = t$ where t varies from 1 to 0.

$$\therefore M = t^2 + t^2 = 2t^2 \;\&\; N = t^2$$

$$dx = dt \;\&\; dy = dt$$

$$\int_C (xy + y^2)dx + x^2dy = \int_1^0 (2t^2)dt + t^2dt = \int_1^0 (2\,t^2 + t^2)dt$$

$$= \left[\frac{3t^3}{3}\right]_1^0 = -1.$$

Hence,

$$\int_C (xy + y^2)dx + x^2dy = \int_{OAB} (xy + y^2)dx + x^2dy + \int_{BDO} (xy + y^2)dx + x^2dy$$

$$= \frac{19}{20} - 1 = -\frac{1}{20} \quad \text{..............................} (1)$$

$$\frac{\partial N}{\partial x} = 2x \;\&\; \frac{\partial M}{\partial y} = x + 2y$$

x varies from 0 to 1 and y varies from $x^2\ to\ x$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^1 \int_{x^2}^{x} (2x - x - 2y)dydx$$

$$= \int_0^1 \int_{x^2}^{x} (x - 2y)dydx = \int_0^1 \left(xy - 2\frac{y^2}{2}\right)_{x^2}^{x} dx$$

$$= \int_0^1 [(xx - x^2) - (x^3 - x^4)]dx = \int_0^1 (x^4 - x^3)dx$$

$$= \left(\frac{x^5}{5} - \frac{x^4}{4}\right)_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} = -\frac{1}{20} \text{ ....} (2)$$

From (1) and (2)

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Hence, Green's theorem is verified.

Problem 2 : Verify Green's theorem in plane for the integral $\int_C x^2 dx + ydy$, where C is the curve enclosing the region R bounded by the parabola $y^2 = x$ and the line $y = x$.

(Hint : Common point (0,0) , (1,1). For the line segment x=t, y=t & t varies from 0 to 1. For the parabolic arc $x = t^2$ & $y = t$, where t varies from 1 to 0. Ans. -1/28).

Problem 3 : Verify Green's theorem in plane for the integral $\int_C x^2 dx + xydy$, where C is the curve enclosing the region R bounded by the parabola $y^2 = 8x$ and the line $y = 2x$.

(Hint : Common point (0,0) , (2,4). For the line segment x=t, y=2t & t varies from 0 to 2. For the parabolic arc $x = 2t^2$ & $y = 4t$, where t varies from 1 to 0. Ans. 8/3)

Problem 4 : Verify Green's theorem for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$, where C is the boundary of the region R enclosed by $y = x^2$ & $y^2 = x$.

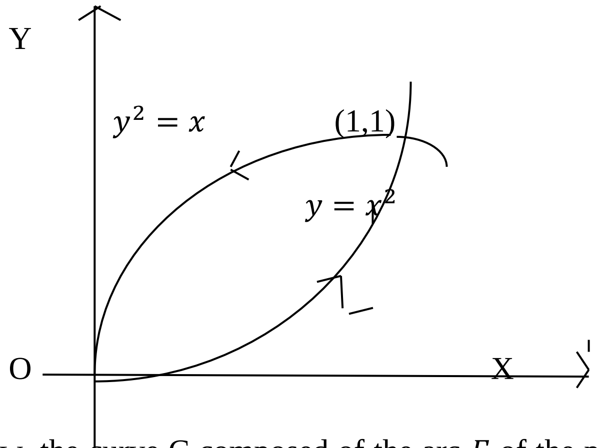
Solution: Given parabolas are $y = x^2$ & $y^2 = x$.

$\therefore x^4 = x => x^4 - x = 0 => x(x^3 - 1) = 0 => x = 0\ or\ x = 1.$

When x=0 , y=0.

When x=1 , y=1.

Let the parabolas intersect at (0,0) and (1,1).



Now, the curve C composed of the arc Γ of the parabola $y = x^2$ and the arc Γ' of the parabola $y^2 = x$.

The parametric equation of Γ is $x = t, y = t^2$, where t varies from 0 to 1.

$$\int_{\Gamma} = \int_0^1 (3t^2 - 8t^4)dt + (4t^2 - 6t^3)(2tdt) = -1 \ (verify)$$

The parametric equation of Γ' is $x = t^2$ & $y = t$, where t varies from 1 to 0.

$$\int_{\Gamma'} = \int_1^0 (3t^4 - 8t^2)(2tdt) + (4t - 6t^3)(dt) = \frac{5}{2} \ (verify)$$

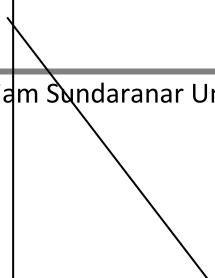
Thus $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = -1 + \frac{5}{2} = \frac{3}{2} \ldots\ldots\ldots\ldots\ldots\ldots.(1)$

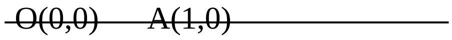
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^1 \int_{x^2}^{\sqrt{x}} 10ydydx = \frac{3}{2} (verify) \ldots \ldots..(2)$$

From (1) & (2) Green's theorem is verified.

Problem 4 ***Verify Green's theorem for*** $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$, ***where C is the boundary of the region R enclosed by x=0, y=0, x+y=1.***

Solution : B(0,1)





$$\int_C = \int_{OA} + \int_{AB} + \int_{BO}$$

Along OA : x=t, y=0, t varies from 0 to 1.

$$\int_{OA}(3x^2 - 8y^2)dx + (4y - 6xy)dy = \int_0^1 3t^2 dt = 1\ (verify)$$

Along AB :

$$\frac{x-1}{0-1} = \frac{y-0}{1-0} = t \Rightarrow \frac{x-1}{-1} = t \Rightarrow x - 1 = -t \Rightarrow x = 1 - t \,\&\, \frac{y}{1} = t \Rightarrow y = t$$

$x = 1 - t, y = t$, t varies from 0 to 1.

$$\int_{AB}(3x^2 - 8y^2)dx + (4y - 6xy)dy = \int_0^1 (-3 + 4t + 11t^2)dt = 8/3\ (verify)$$

Along BO : $x = 0, y = 1 - t$, t varies from 0 to 1.

$$\int_{BO}(3x^2 - 8y^2)dx + (4y - 6xy)dy = \int_0^1 4(t-1)dt = -2\ (verify)$$

Thus, $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = 1 + \frac{8}{3} - 2 = \frac{5}{3}(verify)$ .... (1)

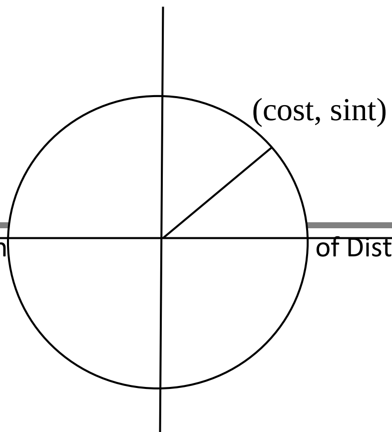
Find $\frac{\partial N}{\partial x}$ & $\frac{\partial M}{\partial y}$

Then, $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^1 \int_0^{1-x} (-6y + 16y)dydx = \frac{5}{3}(verify)$.... (2)

From (1) & (2), Green's theorem is verified.

Problem 5 : Verify Green's theorem for $\int_C (x - 2y)dx + xdy$, where C is the circle $x^2 + y^2 = 1$.

Solution :



The parametric equations of the circle are x=cost, y=sint, t varies from 0 $to\ 2\pi$.

$$dx = -sint\ dt\ \&\ dy\ =\ cost\ dt$$

$$\int_C (x - 2y)dx + xdy = \int_0^{2\pi} (cost - 2sint)(-sintdt) + costcostdt$$

$$= \int_0^{2\pi} (-costsint + 2sin^2t + cos^2t)dy$$

$$= \int_0^{2\pi} \left(-\frac{sin2t}{2} + 2sin^2t + cos^2t\right) dy = 3\pi\ (verifty)$$

Problem 6 : Evaluate $\int_C (3x + 4y)dx + (2x - 3y)dy$, ***where C is the circle*** $x^2 + y^2 = 4$

Problem 7 : Show that $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = \frac{5}{3}$, ***where C is the boundary of the rectangular area enclosed by the lines y=0, x+y=1, x=0.***

Problem 8 : Show that $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = 20$, ***where C is the boundary of the rectangular area enclosed by the lines x=0, x=1, y=0, y=2.***

Problem 9 : Evaluate $\int_C xy^2dy - x^2ydx$, ***where C is the cardioids*** $r = a(1 + cos\theta)$.

[Hint:

$\int_C xy^2dx - x^2ydy = \iint_R (x^2 + y^2)dxdy = \int_0^{2\pi} \int_0^{a(1+cos\theta)} r^2(rdr)d\theta\ = \frac{35}{16}\pi a^4$].

5.3 Stoke's Theorem

If S is the surface bounded by a simple closed curve C then $\oint_C A \cdot dr = \iint_S (\nabla \times A) \cdot ndS$, where A has a continuous derivatives on S.

Problem 1: Verify Stokes theorem for $\vec{A} = (2x - y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ taken over the upper half surface of the sphere $x^2 + y^2 + z^2 = 1, z \geq 0$ and the boundary curve C the circle $x^2 + y^2 = 1, z = 0$.

Solution:

Stoke's theorem is $\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot n dS$

Given, $\vec{A} = (2x - y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$

Put $x = cos\theta, y = sin\theta, z = 0$

$$\therefore \vec{A} = (2cos\theta - sin\theta)\vec{i}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = cos\theta\vec{i} + sin\theta\vec{j}$$

$$d\vec{r} = -sin\theta\vec{i} + cos\theta\vec{j}$$

$$\vec{A} \cdot d\vec{r} = \big((2cos\theta - sin\theta)\vec{i}\big) \cdot (-sin\theta\vec{i} + cos\theta\vec{j})$$

$$= (2cos\theta - sin\theta)(-sin\theta)$$

$$= -2sin\theta cos\theta + sin^2\theta$$

$$\oint_C \vec{A} \cdot d\vec{r} = \int_0^{2\pi} (-2sin\theta cos\theta + sin^2\theta)d\theta$$

$$= \int_0^{2\pi} (-sin2\theta + \frac{1 - cos2\theta}{2})\, d\theta$$

$$= \left[\frac{cos2\theta}{2} + \frac{\theta}{2} - \frac{sin2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{cos4\pi}{2} + \frac{2\pi}{2} - \frac{sin4\pi}{2} - \frac{cos0}{2} + \frac{0}{2} - \frac{sin0}{2}$$

$$= \frac{1}{2} + \pi - 0 - \frac{1}{2} + 0 - 0$$

$= \pi$ ………………… (1)

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(-y^2z) + \frac{\partial}{\partial z}(yz^2)\right) - \vec{j}\left(\frac{\partial}{\partial x}(-y^2z) - \frac{\partial}{\partial z}(2x - y)\right)$$

$$+ \vec{k}\left(\frac{\partial}{\partial x}(-y^2z) - \frac{\partial}{\partial y}(2x - y)\right)$$

$$= \vec{i}(-2yz + 2yz) - \vec{j}(0 - 0) + \vec{k}(0 + 1)$$

$$= 0\vec{i} - 0\vec{j} + 1\vec{k} = \vec{k}$$

$$\vec{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|}$$

$$\emptyset = x^2 + y^2 + z^2 - 1$$

$$\nabla\emptyset = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$|\nabla\emptyset| = \sqrt{4(x^2 + y^2 + z^2)} = 2$$

$$\vec{n} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{2} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\iint_S (\nabla \times \vec{A}) \cdot n dS = \iint_S \vec{k} \cdot n \frac{dxdy}{n \cdot \vec{k}}$$

$$= \iint_S dxdy$$

$$= Area\ of\ the\ sphere$$

$$= \pi r^2$$

$$= \pi\ [since, r = 1\ ] \quad \ldots\ldots\ldots\ldots.(2)$$

From (1) & (2) Stoke's theorem is verified.

Problem 2: Verify Stokes theorem for $\vec{A} = x^2\vec{i} + xy\vec{j}$ taken over the square surface S in the XOY plane whose vertices are O(0,0,0), A(a,0,0), B(a,a,0), C(0,a,0) and over its boundary.

Solution:

Stoke's theorem is $\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot ndS$

Given, $\vec{A} = x^2\vec{i} + xy\vec{j}$

$$\vec{O} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\vec{A} = a\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\vec{B} = a\vec{i} + a\vec{j} + 0\vec{k}$$

$$\vec{C} = 0\vec{i} + a\vec{j} + 0\vec{k}$$

We know that the parametric equation of the points whose position vectors are $\vec{a}$ and $\vec{b}$ are :

$$\vec{r} = (1-t)\vec{a} + t\vec{b}\,, 0 \leq t \leq 1.$$

The parametric equation of OA is $\vec{r} = (1-t)\vec{0} + t(a\vec{\imath})$

$$\vec{r} = at\,\vec{\imath}$$

The parametric equation of AB is $\vec{r} = (1-t)a\vec{\imath} + t(a\vec{\imath} + a\vec{\jmath})$

$$\vec{r} = a\,\vec{\imath} + at\vec{\jmath}$$

The parametric equation of BC is $\vec{r} = (1-t)a\vec{\imath} + a\vec{\jmath}$

The parametric equation of CO is $\vec{r} = (1-t)a\vec{\jmath} + t\vec{0}$

$$\vec{r} = a(1-t)\,\vec{\jmath}$$

$$Now, \oint_{OA} \vec{A}\cdot d\vec{r} = \int_0^1 a^2t^2\vec{\imath}\cdot(adt\vec{\imath})$$

$$= \int_0^1 a^3t^2\,dt$$

$$= a^3\left[\frac{t^3}{3}\right]_0^1$$

$$= \frac{a^3}{3}.$$

$$\oint_{AB} \vec{A}\cdot d\vec{r} = \int_0^1 (a\,\vec{\imath} + at\vec{\jmath})\vec{\imath}\cdot(adt\vec{\jmath})$$

$$= \int_0^1 a^3t\,dt$$

$$= a^3\left[\frac{t^2}{2}\right]_0^1$$

$$= \frac{a^3}{2}.$$

$$\oint_{BC} A\cdot d\vec{r} = \int_0^1 -a^3(1-t)^2dt$$

$$= -a^3 \int_0^1 (1-t)^2\, dt$$

$$= -a^3 \left[t - \frac{2t^2}{2} + \frac{t^3}{3} \right]_0^1$$

$$= -\frac{a^3}{3}.$$

$$\oint_{CO} \vec{A} \cdot dr = 0$$

$$\oint \vec{A} \cdot d\vec{r} = \oint_{OA} \vec{A} \cdot d\vec{r} + \oint_{AB} \vec{A} \cdot d\vec{r} + \oint_{BC} \vec{A} \cdot d\vec{r} + \oint_{CO} \vec{A} \cdot d\vec{r}$$

$$= \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3}$$

$$= \frac{a^3}{2} \ldots \ldots \ldots \ldots \ldots \ldots (1)$$

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= 0\vec{i} - 0\vec{j} + \vec{k}y$$

$$= y\vec{k}$$

$$dS = \frac{dxdy}{n \cdot \vec{k}}$$

$$\iint_S (\nabla \times \vec{A}) \cdot ndS = \int_0^a \int_0^a y\vec{k} \cdot n \, \frac{dxdy}{n \cdot \vec{k}}$$

$$= \int_0^a \int_0^a y \, dxdy$$

$$= \int_0^a [xy]_0^a dy$$

$$= a \left[\frac{y^2}{2} \right]_0^a$$

$$= \frac{a^3}{2} \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

From (1) & (2) Stoke's theorem is verified.

Problem 3: By using Stoke's theorem show that $\nabla \times (\nabla \emptyset) = \vec{0}$.

Solution : Let C b any closed curve and S any surface spanning it. Let $\vec{A} = \nabla \emptyset$.

Stoke's theorem is $\oint_C \vec{A} \cdot dr = \iint_S (\nabla \times \vec{A}) \cdot \vec{n} dS$

$$\iint_S (\nabla \times \vec{A}) \cdot \vec{n} dS = \oint_C \nabla \emptyset \cdot dr$$

$$= \oint_C \left(\frac{\partial \emptyset}{\partial x} \vec{i} + \frac{\partial \emptyset}{\partial y} \vec{j} + \frac{\partial \emptyset}{\partial z} \vec{k} \right) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \oint_C \frac{\partial \emptyset}{\partial x} dx + \frac{\partial \emptyset}{\partial y} dy + \frac{\partial \emptyset}{\partial z} dz$$

$$= \int_C d\emptyset$$

$$= \vec{0}.$$

Exercise :

1. Verify Stokes theorem for $\vec{A} = y\vec{i} + 2yz\vec{j} + y^2\vec{k}$ taken over the upper half surface of the sphere $x^2 + y^2 + z^2 = 1, z \geq 0$ and the boundary curve C the circle $x^2 + y^2 = 1, z = 0$. [Ans. $-\pi$]
2. Verify Stokes theorem for $\vec{A} = y\vec{i} + z\vec{j} + x\vec{k}$ for the upper half of the sphere $x^2 + y^2 + z^2 = 1, z \geq 0$ and the boundary curve C the circle $x^2 + y^2 = 1, z = 0$. [Ans. $-\pi$]
3. Verify Stoke's theorem for $\vec{A} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ taken over the triangular surface S in the plane $x + y + z = 1$ bounded by the planes $x = 0, y = 0, z = 0$ over its boundary. [Hint : The vertices of the triangular surface are A(1,0,0), B(0,1,0), C(0,0,1). Ans. $-\frac{1}{2}$]
4. Verify Stoke's theorem for $\vec{A} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - zx\vec{k}$, where S is the surface of the cube $x = 0, x = 2, y = 0, y = 2, z = 0, z = 2$ above the XOY plane.[Ans. -4]
5. By using Stoke's theorem evaluate the integral $I = \iint_C [(1 + y)z\vec{i} + (1 + z)x\vec{j} + (1 + x)y\vec{k}] \cdot dr$ in the following cases:

(i) C is the circle $x^2 + y^2 = 1, z = 1$ [Hint $\vec{n} = \vec{k}$; $\iint_S 1\, dS = area\ of\ the\ circle = \pi$]

(ii) C is the triangle ABC, where A, B, C are (1,0,0), (0,1,0), (0,0,1). [Ans. 3/2]

(iii) C is a closed curve in the plane $x - 2y + z = 1$. $[Ans. 0]$

6. Evaluate $\iint_S (\nabla \times \vec{A}) \cdot dS$, $if\ \vec{A} = y\vec{i} + z\vec{j} + x\vec{k}$ where S is the paraboloidal surface $x^2 + y^2 = 1 - z, z \geq 0$. [Ans. $-\pi$].

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